H^{∞}-Optimal Control under Imperfect State Measurements using Game Theoretic Approach

Zigang Pan[1*]

[1*]4797 Bordeaux Lane, Mason, 45040, Ohio, USA.

Corresponding author(s). E-mail(s): zigangpan2002@mac.com;

Abstract

This paper studies both the finite-horizon and the infinite-horizon H-infinity optimal control for linear systems under imperfect state measurements using the game theoretic approach. In the finite-horizon case, the result to this problem is well-known that the solution exists under the existences of solutions to two generalized Riccati differential equations and that they satisfy the spectral radius condition. In the existing game theoretic approach to this problem, it calls for the separation principle, which is a deep principle that is not obvious for beginners. This paper offers a completely elementary solution to the problem that uses the completion of squares method and Riccati differential equation solutions. This solution method calls for estimation and control sequential design, and only after the controller has been obtained, do I relate the over all solution to the control Riccati differential equation. In the infinite-horizon case, the standard approaches call for the inner systems, which is something that is not familiar with the control community. I obtain the exact conditions under which the infinite-horizon problem will admit an internally stabilizing solution. This exact set of conditions basically solves the H-infinity optimal control problem completely in conjunction with Theorem 9.8 of Başar and Bernhard (1995). I then obtain another set of exact conditions as a corollary, which is quite standard, under which the infinite-horizon problem will admit an internally stabilizing solution that is the appropriate limit of the finite-horizon solution.

Keywords: H-infinity optimal control, game theoretic approach, completion of squares, generalized Riccati differential equation, generalized algebraic Riccati equation

1 Introduction

H^∞-optimal control had been a topic of intense research in the 1980s. The problem is interesting since it tries to optimize the H^∞-norm of the transfer function matrix for a given linear time-invariant system, which is shown to equal to the L_2-induced norm of the system (Zames, 1981). This induced norm perspective is powerful and allows the small gain theorem to apply for the closed-loop system that renders the H^∞ controller robust to a general class of systems that can be represented as the nominal system in feedback with some L_2-induced norm bounded uncertainty block. The solution to the H^∞-optimal control problem is first obtained in Doyle et al. (1989). Further research along the line has been focused on time-varying systems (Limebeer et al., 1989; Ravi et al., 1991; Khargonekar et al., 1991), mixed H^2 and H^∞ control design (Bernstein and Haddad, 1989; Khargonekar and Rotea, 1991; Stoorvogel, 1993), jump linear systems (de Souza and Fragoso, 1993), singularly perturbed systems (Pan and Başar, 1994, 1996b), sampled-data control systems (Bamieh and Pearson, 1992), and nonlinear systems (van der Schaft, 1991; Isidori and Astolfi, 1992; Toussaint et al., 2000). One of the main research thrust has been the identification of the H^∞ central controller and the optimal control law for linear exponential of quadratic Gaussian (LEQG) problem (Rhee and Speyer, 1989; Uchida and Fujita, 1989; Whittle, 1990; Fleming and McEneaney, 1992; Pan and Başar, 1996a). This equivalence motivated further research into risk-sensitive control of nonlinear systems and the observation that the large deviation limit for nonlinear exponential of integral cost Gaussian problem is identical to the nonlinear zero-sum game solution (James et al., 1993; Runolfsson, 1994; Fleming and McEneaney, 1995; James and Yuliar, 1995; Bensoussan and Elliott, 1996; Moon et al., 2019). A number of books has been written on H^∞ control (Doyle et al., 1992; Stoorvogel, 1992; Başar and Bernhard, 1995; Zhou et al., 1996; Trentelman et al., 2001) to further relax the assumptions on the problem. Noticing the induced-norm implication, a game-theoretic approach to the H^∞-optimal control is quickly adopted to address the problem in the framework of soft-constrained zero-sum games (Didinsky and Başar, 1992; Başar and Bernhard, 1995). This approach allows for investigation of finite-horizon problem for linear time-varying systems, as well as nonlinear systems. The cost-to-come function (Didinsky et al., 1993; Başar and Bernhard, 1995) approach to the problem solves the estimation part of the problem, while the control part of the problem is well-known as the perfect state measurements case with well-established solutions. This then leads to the separation principle (Didinsky et al., 1993; Başar and Bernhard, 1995) that connects these two solutions for a complete solution to the problem that agrees with the well-known solution (Doyle et al., 1989). In the standard approach to the H^∞-optimal control, without using the game theoretic formulation, refined solutions (Zhou et al., 1996; Trentelman et al., 2001) then calls for the inner system for imperfect state measurement case, which is not that familiar with the control community.

This paper is to study both the finite-horizon and the infinite-horizon H^∞-optimal control for linear systems under imperfect state measurements using the game theoretic approach. First, I rederive the finite-horizon H^∞-optimal control for linear time-varying systems under imperfect state measurements using the game theoretic approach. It offers a completely elementary solution to the problem that uses the completion of squares method and Riccati differential equation solutions. This solution method calls for estimation and control sequential design, and only after the controller has been obtained, do I relate the over all solution to the control Riccati differential equation. This solution methodology offers good opportunity to generalize to nonlinear systems, such as the robust adaptive control problems that this author has been working on for many years. After the successful solution to the finite-horizon problem, I then study the infinite-horizon case for linear time-invariant systems. I first show that the generalized algebraic Riccati equation $\bar{A}'Z + Z\bar{A} - Z\bar{S}Z + \bar{Q} = \mathbf{0}$, where $\bar{Q}$ is positive-semidefinite, admits a positive-semidefinite stabilizing solution implies that the pair $(\bar{A}, \bar{Q})$ has no unobservable modes on the imaginary axis. A set of exact conditions is obtained for the infinite-horizon H^∞-optimal control problem under imperfect state measurements such that a solution exists and is further internally stabilizing (i. e., asymptotically stable under the optimal control strategy and without any exogeneous disturbance input.) This exact set of conditions basically solves the H^∞-optimal control problem completely in conjunction with Theorem 9.8 of Başar and Bernhard (1995), which says that the H^∞-optimal control problem admits a stabilizing solution if, and only if, the control and the estimation generalized algebraic Riccati equations admit positive-semidefinite and stablizing solutions that further satisfies the spectral radius condition. I then obtain exact conditions, which is quite standard, such that the solution is the appropriate limit of the finite-horizon solution.

The balance of the paper is as follows. In the next section, I will summarize the notations used in the paper. Then, in Section 3, I will formulate the finite-horizon problem to be studied using a game theoretic approach. In Section 4, the estimation part of the problem is solved by calculating the cost-to-come function for the problem explicitly, which then converts the problem into a H^∞-optimal control problem under full-information measurements. This full-information measurement problem is then solved in Section 5 using standard completion of square method with a Riccati differential equation that is not a familiar one in the well-known solution to the problem. In Section 6, I will then show that the solution to this Riccati differential equation is nothing but a nonlinear function of the solutions to the two well-known Riccati differential equations that also reveal the spectral radius condition. Then, the well-known solution is recovered completely in finite-horizon case. In Section 7, I further study the infinite-horizon H^∞-optimal control for linear time-invariant systems under imperfect state measurements. I obtain a set of exact conditions for the infinite-horizon problem such that a solution exists and the closed-loop system is internally stable. Then, I prescribe another set of exact conditions under which the optimal control strategy is the appropriate limit of the optimal strategy for the finite-horizon case as $t_0 \downarrow -\infty$ and $t_f \uparrow \infty$. An example is included to illustrate the theoretical results in Section 8. The paper ends with some concluding remarks in Section 9.

2 Notations

Let $\mathbb{R}$ denote the real line; $\mathbb{R}_+ := (0, \infty) \subset \mathbb{R}$; $\overline{\mathbb{R}_+} := [0, \infty) \subset \mathbb{R}$; $\mathbb{C}$ denote the complex plane; $\mathbb{C}_+$ and $\mathbb{C}_-$ denote the open right half and open left half of the complex plane, respectively; $\overline{\mathbb{C}_+}$ and $\overline{\mathbb{C}_-}$ denote the close right half and the closed left half of the complex plane, respectively; i denotes the complex unit; $\forall a \in \mathbb{C}$, $\overline{a}$ denote the complex conjugate of a; $\mathbb{N} := \{1, 2, \dots\}$; $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Unless specified, all signals, constants, and matrices are real. $\mathcal{C}$ means continuous and $\mathcal{C}_k$ means continuous up to kth derivative; $\frac{\partial}{\partial x}$ denotes partial derivative with respect to x. For a $\mathcal{B}_\mathrm{B}(\mathbb{R})$-measurable function (Pan, 2024) $f : I \to \mathbb{R}^n$ on an interval $I \subseteq \mathbb{R}$, $f \in \bar{\mathrm{L}}_p$, if $(\int_I |f(\tau)|^p \,\mathrm{d}\tau)^{1/p} < \infty$. $\forall z \in \mathbb{R}^n$, $|z| := \sqrt{z'z}$; $|z|_Q^2 := z'Qz$, $\forall Q \in \mathcal{S}_n$. $\mathcal{S}_n$, $\mathcal{S}_{\mathrm{psd}\, n}$, and $\mathcal{S}_{+\, n}$ denote $n \times n$-dimensional real symmetric, positive-semidefinite, and positive-definite matrices, respectively; $Q_1 \leq Q_2$, if $Q_2 - Q_1 \in \mathcal{S}_{\mathrm{psd}\, n}$; $Q_1 < Q_2$, if $Q_2 - Q_1 \in \mathcal{S}_{+\, n}$; $\forall Q_1, Q_2 \in \mathcal{S}_n$. I_n denotes the $n \times n$-dimensional identity matrix. $\mathbf{0}_{m\times n}$ denotes the $m \times n$-dimensional matrix whose elements are all zeros. $\star$ denote some matrix of appropriate dimensions that is not of any particular interest. I will also drop the subscript of I_n and/or $\mathbf{0}_{m\times n}$ to denote the identity matrix and/or the completely zero matrix, whose dimensions are clear from the context, respectively.

3 Problem formulation

Consider a continuous-time linear time-varying system:

$$\dot{x} = Ax + Bu + Dw; \quad x(t_0) = x_0 \in \mathbb{R}^n \tag{1a}$$

$$y = Cx + Ew \tag{1b}$$

$$z = Hx + Gu \tag{1c}$$

where x is n-dimensional state, $n \in \mathbb{Z}_+$; $x_0 \in \mathbb{R}^n$ is the unknown initial state variable with initial guess $\check{x}_0 = \mathbf{0}_n \in \mathbb{R}^n$; u is p-dimensional control input, $p \in \mathbb{Z}_+$; w is q-dimensional disturbance input, $q \in \mathbb{Z}_+$; y is m-dimensional measurement output, $m \in \mathbb{Z}_+$; z is k-dimensional controlled output, $k \in \mathbb{Z}_+$; and A, B, D, C, E, H, and G are matrix-valued functions of time, are $\mathcal{B}_\mathrm{B}(\mathbb{R})$-measurable and bounded, and are of appropriate dimensions. Under consideration is the finite horizon problem with time interval $[t_0, t_f] \subset \mathbb{R}$. The H^∞-optimal control problem is to find $\gamma^* \in \overline{\mathbb{R}_+}$ such that $\forall \gamma > \gamma^*$, there exists a causal controller $u(t) = \mu(t, \check{x}_0, y_{[t_0,t]})$, $\forall t \in [t_0, t_f] \subset \mathbb{R}$ among the set of admissible controllers $\mathcal{M}$, such that the L_2-induced norm from the disturbance input w and the initial state x_0 to the controlled output z and the terminal state $x(t_f)$ is less than or equal to γ, i. e.,

$$\int_{t_0}^{t_f} |z(\tau)|^2 \,\mathrm{d}\tau + |x(t_f)|_{Q_f}^2 \leq \gamma^2 \left(\int_0^{t_f} |w(\tau)|^2 \,\mathrm{d}\tau + |x_0|_{Q_0}^2 \right) \tag{2}$$

$\forall w_{[t_0,t_f]} \in \bar{\mathrm{L}}_2([t_0, t_f], \mathbb{R}^q)$, where $Q_0 \in \mathcal{S}_{+\, n}$ and $Q_f \in \mathcal{S}_{\mathrm{psd}\, n}$ are given initial and terminal weighting matrices; and $\forall \gamma < \gamma^*$, no such causal controller exists. The set of admissible controllers is defined to be (in the finite-horizon case)

$$\mathcal{M} := \Big\{ \mu : [t_0, t_f] \times \mathbb{R}^n \times \bar{\mathrm{L}}_2([t_0, t_f], \mathbb{R}^m) \to \mathbb{R}^p \;\Big|\; \mu \text{ is a causal controller and } \forall \check{x}_0 \in \mathbb{R}^n, \forall y_{[t_0,t_f]} \in \bar{\mathrm{L}}_2([t_0, t_f], \mathbb{R}^m), \text{ we have } u_{[t_0,t_f]} \in \bar{\mathrm{L}}_2([t_0, t_f], \mathbb{R}^p). \Big\}$$

The game theoretic approach to the above problem is to form the soft-constrained cost function

$$\bar{J}_\gamma(\mu, \nu, Q_0, Q_f) := \int_{t_0}^{t_f} (|z(\tau)|^2 - \gamma^2 |w(\tau)|^2) \,\mathrm{d}\tau + |x(t_f)|_{Q_f}^2 - \gamma^2 |x_0|_{Q_0}^2 \tag{3}$$

and investigate the upper value of the game $\inf_\mu \sup_\nu \bar{J}_\gamma(\mu, \nu, Q_0, Q_f) \le 0$, where ν is the maximizer's strategy of choice for $x_0 \in \mathbb{R}^n$ and $w_{[t_0,t_f]} \in \bar{\mathrm{L}}_2([t_0, t_f], \mathbb{R}^q)$. It is easy to see that γ^* corresponds to the infimum of set $\Gamma := \{\gamma \in \mathbb{R}_+ \mid \inf_\mu \sup_\nu \bar{J}_\gamma(\mu, \nu, Q_0, Q_f) \le 0\}$. In the place of the H^∞-optimal control problem, we are now focused on solving the upper values of a class of soft-constrained zero-sum differential game parametrized by the performance level $\gamma \in \mathbb{R}_+$. When this upper value is less than or equal to zero, than, $\gamma \in \Gamma$ and $\gamma^* \le \gamma$. On the other hand, when the upper value is positive, which will then equal to ∞ by the linear quadratic nature of the problem, then $\gamma \notin \Gamma$ and $\gamma \le \gamma^*$.

This class of parametrized game problem can be solved using linear quadratic optimization if we note the following identity:

$$\begin{aligned}
\inf_\mu \sup_\nu \bar{J}_\gamma(\mu, \nu, Q_0, Q_f) &= \inf_\mu \sup_{x_0 \in \mathbb{R}^n, w_{[t_0,t_f]} \in \bar{\mathrm{L}}_2} \bar{J}_\gamma(\mu, \nu, Q_0, Q_f) \\
&= \inf_\mu \sup_{x(t_f) \in \mathbb{R}^n, y_{[t_0,t_f]} \in \bar{\mathrm{L}}_2} \sup_{x_0 \in \mathbb{R}^n, w_{[t_0,t_f]} \in \bar{\mathrm{L}}_2 | x(t_f), y_{[t_0,t_f]}} \bar{J}_\gamma(\mu, \nu, Q_0, Q_f)
\end{aligned} \tag{4}$$

where the first equality follows since we are dealing with the upper value of the game, and the second equality follows since this is a team optimization problem for the suprema in the second equality, and it doesn't matter how suprema are calculated (see Proposition 8.36 of Pan (2024)).

The inner most supremization is over all initial condition and all disturbance waveform that generate the terminal state $x(t_f)$ and the measurement output waveform $y_{[t_0,t_f]}$. Since the controller is a causal function of $y_{[t_0,t_f]}$, then the control input is an open-loop time function once the measurement output waveform $y_{[t_0,t_f]}$ is fixed. This supremization can be carried out as a single person linear quadratic optimization problem. This step corresponds to the estimator design step and is precisely the calculation of the cost-to-come function (Didinsky et al., 1993) for the problem. This step will be covered in the next section.

Once the cost-to-come function is calculated, which will assume a general quadratic structure, then the problem is converted into a zero-sum full-information differential game problem, that can be solved using standard completion-of-squares method. The Riccati differential equation that is associated with this full-information game problem is not familiar in general, and may suggest that the existence condition of the Riccati equation will depend on the solution to the estimation Riccati differential equation. This step is the control design step, which will be covered in Section 5.

In Section 6, we explicitly solve the Riccati differential equation for the full-information game, and show that the solution is a function of the two well-known (estimation and control) Riccati differential equations with the spectral radius condition built-in. Thus, we recover the well-known solution to the H^∞-optimal control problem.

To allow generality in the study, we will try to compute the upper value of the following zero-sum differential game with the soft-constrained cost function:

$$J_\gamma(\mu, \nu, \check{x}_0, Q_0, Q_f) := \int_{t_0}^{t_f} (|z(\tau)|^2 - \gamma^2 |w(\tau)|^2) \, \mathrm{d}\tau + |x(t_f)|^2_{Q_f} - \gamma^2 |x_0 - \check{x}_0|^2_{Q_0} \quad (5)$$

where we have accommodated the initial guess $\check{x}_0 \in \mathbb{R}^n$ into the cost. It should be easy to see that such a change is compatible with our solution methodology. The original upper value is simply the upper value for the modified cost function with $\check{x}_0$ set to $\mathbf{0}_n$.

Now, we turn to the estimation design step in the next section.

4 Estimation design

In this section, we will compute the cost-to-come function explicitly, which is the calculation of the innermost supremization in (4):

$$\begin{aligned} &\sup_{x_0 \in \mathbb{R}^n, w_{[t_0,t_f]} \in \bar{\mathbb{L}}_2 | x(t_f), y_{[t_0,t_f]}} J_\gamma(\mu, \nu, \check{x}_0, Q_0, Q_f) \\ &=: W_\gamma(t_f, x(t_f), u_{[t_0,t_f]}, y_{[t_0,t_f]}) + |\, x(t_f)\,|^2_{Q_f} \end{aligned}$$

This is a one-person linear quadratic optimization problem. To avoid singularity, it is assumed that

Assumption 1. $E(t)(E(t))' := N(t) \in \mathcal{S}_{+\,m}$, $\forall t \in [t_0, t_f] \subset \mathbb{R}$.

We will seek a causal solution in quadratic form:

$$W_\gamma(t, x, u_{[t_0,t]}, y_{[t_0,t]}) = -\,|\, x - \lambda(t)\,|^2_{(\Sigma(t))^{-1}} + \eta(t)$$

Then, the initial condition at $t = t_0$ for W_γ is

$$W_\gamma(t_0, x_0, u_{[t_0,t_0]}, y_{[t_0,t_0]}) = -\,|\, x_0 - \check{x}_0\,|^2_{\gamma^2 Q_0}$$

and the cost-to-come function W_γ further satisfies the partial differential equation:

$$\begin{aligned} &\sup_{w(t)|y(t)=Cx(t)+Ew(t)} \Big(-\frac{\partial W_\gamma}{\partial t} - \frac{\partial W_\gamma}{\partial x}(A(t)x(t) + B(t)u(t) + D(t)w(t)) \\ &+(H(t)x(t) + G(t)u(t))'(H(t)x(t) + G(t)u(t)) - \gamma^2 (w(t))'w(t) \Big) = 0 \\ \Leftrightarrow\ & -\frac{\partial W_\gamma}{\partial t} - \frac{\partial W_\gamma}{\partial x}(A(t)x(t) + B(t)u(t)) + (H(t)x(t) + G(t)u(t))'(H(t)x(t) \\ &+G(t)u(t)) - \gamma^2 \left| y(t) - C(t)x(t) + \frac{1}{2\gamma^2} E(t)(D(t))' \Big(\frac{\partial W_\gamma}{\partial x}\Big)' \right|^2_{(N(t))^{-1}} \end{aligned}$$

$$+\frac{1}{4\gamma^2}\frac{\partial W_\gamma}{\partial x}D(t)(D(t))'\left(\frac{\partial W_\gamma}{\partial x}\right)' = 0 \tag{6}$$

where the worst-case disturbance input is (applying Lemma 5 of Pan and Başar (2000))

$$\begin{aligned} w_{op}(t) &= (E(t))'(N(t))^{-1}(y(t)-C(t)x(t)) - \frac{1}{2\gamma^2}(I_q - (E(t))'(N(t))^{-1}E(t)) \\ &\quad \cdot (D(t))'\left(\frac{\partial W_\gamma}{\partial x}\right)' \end{aligned} \tag{7}$$

Here, the derivation follows closely to the derivation in Appendix B of Pan and Başar (2000). We will just summarize the result here.

$$\begin{aligned} Q(t) &:= (H(t))'H(t);\ P(t) := (H(t))'G(t);\ R(t) := (G(t))'G(t);\ L(t) := D(t)(E(t))' \\ \dot{\Sigma}(t) &= (A(t) - L(t)(N(t))^{-1}C(t))\Sigma(t) + \Sigma(t)(A(t) - L(t)(N(t))^{-1}C(t))' \\ &\quad -\Sigma(t)(\gamma^2(C(t))'(N(t))^{-1}C(t) - Q(t))\Sigma(t) + \gamma^{-2}D(t)(D(t))' \\ &\quad -\gamma^{-2}L(t)(N(t))^{-1}(L(t))'; \quad \Sigma(t_0) = \gamma^{-2}Q_0^{-1} \end{aligned} \tag{8a}$$

$$\begin{aligned} \dot{\lambda}(t) &= (A(t) + \Sigma(t)Q(t))\lambda + B(t)u(t) + \Sigma(t)P(t)u(t) + (\gamma^2\Sigma(t)(C(t))' \\ &\quad +L(t))(N(t))^{-1}(y(t) - C(t)\lambda(t)); \quad \lambda(t_0) = \check{x}_0 \end{aligned} \tag{8b}$$

$$\begin{aligned} \dot{\eta}(t) &= (H(t)\lambda(t) + G(t)u(t))'(H(t)\lambda(t) + G(t)u(t)) \\ &\quad -\gamma^2\,|y(t) - C(t)\lambda(t)|^2_{(N(t))^{-1}}; \quad \eta(t_0) = 0 \end{aligned} \tag{8c}$$

The inner supremum has a finite value if there exists a positive definite solution Σ on the interval $[t_0, t_f]$. We will define $\bar{\Sigma} := \gamma^2\Sigma$, then $\bar{\Sigma}$ satisfies the well-known estimation Riccati differential equation

$$\begin{aligned} \dot{\bar{\Sigma}}(t) &= (A(t) - L(t)(N(t))^{-1}C(t))\bar{\Sigma}(t) + \bar{\Sigma}(t)(A(t) - L(t)(N(t))^{-1}C(t))' \\ &\quad -\bar{\Sigma}(t)((C(t))'(N(t))^{-1}C(t) - \gamma^{-2}Q(t))\bar{\Sigma}(t) + D(t)(D(t))' \\ &\quad -L(t)(N(t))^{-1}(L(t))'; \quad \bar{\Sigma}(t_0) = Q_0^{-1} \end{aligned} \tag{9a}$$

We will also define $\check{x} := \lambda$, which is the estimate of the state x based on the form of W_γ. Then, $\check{x}$ satisfy the following dynamics:

$$\begin{aligned} \dot{\check{x}}(t) &= (A(t) + \gamma^{-2}\bar{\Sigma}(t)Q(t))\check{x} + B(t)u(t) + \gamma^{-2}\bar{\Sigma}(t)P(t)u(t) + (\bar{\Sigma}(t)(C(t))' \\ &\quad +L(t))(N(t))^{-1}(y(t) - C(t)\check{x}(t)); \quad \check{x}(t_0) = \check{x}_0 \end{aligned} \tag{9b}$$

The cost function J_γ, (5), admits the following equivalent form:

$$\begin{aligned} J_\gamma(\mu, \nu, \check{x}_0, Q_0, Q_f) &= |x(t_f)|^2_{Q_f} - \gamma^2\,|x(t_f) - \check{x}(t_f)|^2_{(\bar{\Sigma}(t_f))^{-1}} + \int_{t_0}^{t_f} (\big|H(t)\check{x}(t) \\ &\quad +G(t)u(t)\big|^2 - \gamma^2\,\big|y(t) - C(t)\check{x}(t)\big|^2_{(N(t))^{-1}} - \gamma^2\,|w(t) - w_{op}(t)|^2)\,\mathrm{d}t \end{aligned} \tag{9c}$$

The upper value of the differential game is then equal to

$$\inf_{\mu}\sup_{\nu} J_\gamma(\mu,\nu,\check{x}_0,Q_0,Q_f) = \inf_{\mu}\sup_{x(t_f)\in\mathbb{R}^n, y_{[t_0,t_f]}\in\bar{\mathrm{L}}_2}\Big(W_\gamma(t_f,x(t_f),u_{[t_0,t_f]},y_{[t_0,t_f]})$$
$$+\left|x(t_f)\right|^2_{Q_f}\Big) = \inf_{\mu}\sup_{x(t_f)\in\mathbb{R}^n, y_{[0,t_f]}\in\bar{\mathrm{L}}_2}\Big(\int_{t_0}^{t_f}(\left|H(t)\check{x}(t)+G(t)u(t)\right|^2-\gamma^2\left|y(t)\right.$$
$$\left.-C(t)\check{x}(t)\right|^2_{(N(t))^{-1}})\,\mathrm{d}t+\left|x(t_f)\right|^2_{Q_f}-\gamma^2\left|x(t_f)-\check{x}(t_f)\right|^2_{(\bar{\Sigma}(t_f))^{-1}}\Big) \tag{10}$$

and worst-case disturbance input is

$$\begin{aligned}w_{op}(t) \;=\; &(E(t))'(N(t))^{-1}(y(t)-C(t)x(t))+(I_q-(E(t))'(N(t))^{-1}E(t))\\ &\cdot(D(t))'(\bar{\Sigma}(t))^{-1}(x(t)-\check{x}(t))\end{aligned} \tag{11}$$

We will complete a square for the last two terms in (10) and maximize out the terminal state $x(t_f)$, which is finite if $\gamma^2(\bar{\Sigma}(t_f))^{-1}-Q_f\in\mathcal{S}_{+\,n}$. The worst-case $x(t_f)$ is

$$x(t_f) \;=\; \gamma^2(\gamma^2(\bar{\Sigma}(t_f))^{-1}-Q_f)^{-1}(\bar{\Sigma}(t_f))^{-1}\check{x}(t_f)=(I_n-\gamma^{-2}\bar{\Sigma}(t_f)Q_f)^{-1}\check{x}(t_f) \tag{12}$$

and the upper value is equal to

$$\begin{aligned}&\inf_{\mu}\sup_{\nu} J_\gamma(\mu,\nu,\check{x}_0,Q_0,Q_f)\\
&=\inf_{\mu}\sup_{y_{[t_0,t_f]}\in\bar{\mathrm{L}}_2}\Big(\int_{t_0}^{t_f}(\left|H(t)\check{x}(t)+G(t)u(t)\right|^2-\gamma^2\left|y(t)-C(t)\check{x}(t)\right|^2_{(N(t))^{-1}})\,\mathrm{d}t\\
&\quad+\left|\check{x}(t_f)\right|^2_{\gamma^2(\bar{\Sigma}(t_f))^{-1}(\gamma^2(\bar{\Sigma}(t_f))^{-1}-Q_f)^{-1}\gamma^2(\bar{\Sigma}(t_f))^{-1}-\gamma^2(\bar{\Sigma}(t_f))^{-1}}\Big)\\
&=\inf_{\mu}\sup_{y_{[t_0,t_f]}\in\bar{\mathrm{L}}_2}\Big(\int_{t_0}^{t_f}(\left|H(t)\check{x}(t)+G(t)u(t)\right|^2-\gamma^2\left|y(t)-C(t)\check{x}(t)\right|^2_{(N(t))^{-1}})\,\mathrm{d}t\\
&\quad+\left|\check{x}(t_f)\right|^2_{(I_n-\gamma^{-2}Q_f\bar{\Sigma}(t_f))^{-1}Q_f}\Big)\end{aligned} \tag{13}$$

This now is an upper value calculation of a full-information game with (9b) as the system dynamics and (13) as the soft-constrained game cost function.

An equivalent form of the cost function (5) is given as follows, by (9c).

$$\begin{aligned}&J_\gamma(\mu,\nu,\check{x}_0,Q_0,Q_f)=-\left|x(t_f)-(I_n-\gamma^{-2}\bar{\Sigma}(t_f)Q_f)^{-1}\check{x}(t_f)\right|^2_{\gamma^2(\bar{\Sigma}(t_f))^{-1}-Q_f}\\
&\quad+\left|\check{x}(t_f)\right|^2_{(I_n-\gamma^{-2}Q_f\bar{\Sigma}(t_f))^{-1}Q_f}+\int_{t_0}^{t_f}(\left|H(t)\check{x}(t)+G(t)u(t)\right|^2\\
&\quad-\gamma^2\left|y(t)-C(t)\check{x}(t)\right|^2_{(N(t))^{-1}}-\gamma^2\left|w(t)-(E(t))'(N(t))^{-1}(y(t)-C(t)x(t))\right.\\
&\quad\left.-(I_q-(E(t))'(N(t))^{-1}E(t))(D(t))'(\bar{\Sigma}(t))^{-1}(x(t)-\check{x}(t))\right|^2)\,\mathrm{d}t\end{aligned} \tag{14}$$

Now, we turn to the control design in the next section.

5 Control design

In this section, we will solve the full-information soft-constrained zero-sum differential game problem with (9b) as the state dynamics and (13) as the cost function. Instead of maximizing over $y_{[t_0,t_f]}$, we can equivalently maximize over $v_{[t_0,t_f]}$ since we are interested in the upper value of the game, where $v(t) := y(t) - C(t)\check{x}(t)$, $\forall t \in [t_0, t_f] \subset \mathbb{R}$. In terms of v, we have the dynamics of $\check{x}$ is given by

$$\begin{aligned}\dot{\check{x}}(t) \;=\; & (A(t) + \gamma^{-2}\bar{\Sigma}(t)Q(t))\check{x} + B(t)u(t) + \gamma^{-2}\bar{\Sigma}(t)P(t)u(t) + (\bar{\Sigma}(t)(C(t))' \\ & + L(t))(N(t))^{-1}v(t); \quad \check{x}(t_0) = \check{x}_0 \end{aligned} \tag{15}$$

and the upper value of the game is equal to

$$\begin{aligned}\inf_{\mu}\sup_{\nu} J_\gamma(\mu, \nu, \check{x}_0, Q_0, Q_f) = \inf_{\mu} \sup_{v_{[t_0,t_f]}\in\bar{\mathbb{L}}_2} & \Big(\int_{t_0}^{t_f} (\,|\, H(t)\check{x}(t) + G(t)u(t)\,|^2 \\ -\gamma^2 \,|\, v(t)\,|^2_{(N(t))^{-1}})\,\mathrm{d}t + |\,\check{x}(t_f)\,|^2_{(I_n - \gamma^{-2}Q_f\bar{\Sigma}(t_f))^{-1}Q_f} & \Big)\end{aligned} \tag{16}$$

This is a standard game problem, and we make the following assumption to avoid singularity in the optimization.
Assumption 2. $R(t) = (G(t))'G(t) \in \mathcal{S}_{+\,p}$, $\forall t \in [t_0, t_f] \subset \mathbb{R}$.

Then, the solution to this problem can be obtained if the following Riccati differential equation admits a positive semi-definite solution on the interval $[t_0, t_f]$.

$$\begin{aligned}\mathbf{0}_{n\times n} \;=\; & \dot{\Xi}(t) + \Xi(t)(A(t) + \gamma^{-2}\bar{\Sigma}(t)Q(t) - (B(t) + \gamma^{-2}\bar{\Sigma}(t)P(t))(R(t))^{-1}(P(t))') \\ & + (A(t) + \gamma^{-2}\bar{\Sigma}(t)Q(t) - (B(t) + \gamma^{-2}\bar{\Sigma}(t)P(t))(R(t))^{-1}(P(t))')'\Xi(t) \\ & - \Xi(t)((B(t) + \gamma^{-2}\bar{\Sigma}(t)P(t))(R(t))^{-1}(B(t) + \gamma^{-2}\bar{\Sigma}(t)P(t))' \\ & - \gamma^{-2}(\bar{\Sigma}(t)(C(t))' + L(t))(N(t))^{-1}(\bar{\Sigma}(t)(C(t))' + L(t))')\Xi(t) \\ & + Q(t) - P(t)(R(t))^{-1}(P(t))'; \quad \Xi(t_f) = (I_n - \gamma^{-2}Q_f\bar{\Sigma}(t_f))^{-1}Q_f \end{aligned} \tag{17a}$$

where the optimal control law is

$$u(t) = \mu_{\text{opt}}(t, \check{x}_0, y_{[t_0,t]}) = -(R(t))^{-1}((B(t) + \gamma^{-2}\bar{\Sigma}(t)P(t))'\Xi(t) + (P(t))')\check{x}(t) \tag{17b}$$

and the worst case measurement waveform is

$$y(t) = C(t)\check{x}(t) + \gamma^{-2}(C(t)\bar{\Sigma}(t) + (L(t))')\Xi(t)\check{x}(t) \tag{17c}$$

and the upper value of the game is

$$\inf_{\mu}\sup_{\nu} J_\gamma(\mu, \nu, \check{x}_0, Q_0, Q_f) = |\,\check{x}_0\,|^2_{\Xi(t_0)} \tag{18}$$

Clearly, the upper value is zero if $\check{x}_0 = \mathbf{0}_n$.

Using the solution Ξ, we can obtain the following equivalent form of the cost function $J_\gamma(\mu, \nu, \check{x}_0, Q_0, Q_f)$ (5),

$$\begin{aligned}
J_\gamma(\mu, \nu, \check{x}_0, Q_0, Q_f) &= -\left| x(t_f) - (I_n - \gamma^{-2}\bar{\Sigma}(t_f)Q_f)^{-1}\check{x}(t_f) \right|^2_{\gamma^2(\bar{\Sigma}(t_f))^{-1}-Q_f} \\
&\quad + |\check{x}_0|^2_{\Xi(t_0)} + \int_{t_0}^{t_f} \Big(\left| u(t) + (R(t))^{-1}((B(t) + \gamma^{-2}\bar{\Sigma}(t)P(t))'\Xi(t) + (P(t))')\check{x}(t) \right|^2_{R(t)} \\
&\quad -\gamma^2 \left| y(t) - C(t)\check{x}(t) - \gamma^{-2}(C(t)\bar{\Sigma}(t) + (L(t))')\Xi(t)\check{x}(t) \right|^2_{(N(t))^{-1}} \\
&\quad -\gamma^2 \Big| w(t) - (E(t))'(N(t))^{-1}(y(t) - C(t)x(t)) - (I_q - (E(t))'(N(t))^{-1}E(t))(D(t))' \\
&\quad \cdot(\bar{\Sigma}(t))^{-1}(x(t) - \check{x}(t)) \Big|^2 \Big) \,\mathrm{d}t \\
&= -\left| x(t_f) - (I_n - \gamma^{-2}\bar{\Sigma}(t_f)Q_f)^{-1}\check{x}(t_f) \right|^2_{\gamma^2(\bar{\Sigma}(t_f))^{-1}-Q_f} \\
&\quad + |\check{x}_0|^2_{\Xi(t_0)} + \int_{t_0}^{t_f} \Big(\left| u(t) + (R(t))^{-1}((B(t) + \gamma^{-2}\bar{\Sigma}(t)P(t))'\Xi(t) + (P(t))')\check{x}(t) \right|^2_{R(t)} \\
&\quad -\gamma^2 \Big| w(t) - (I_q - (E(t))'(N(t))^{-1}E(t))(D(t))'(\bar{\Sigma}(t))^{-1}(x(t) - \check{x}(t)) \\
&\quad +(E(t))'(N(t))^{-1}(C(t)(x(t) - \check{x}(t)) - \gamma^{-2}(C(t)\bar{\Sigma}(t) + (L(t))')\Xi(t)\check{x}(t)) \Big|^2 \Big) \,\mathrm{d}t \quad (19)
\end{aligned}$$

Therefore, the worst case disturbance waveform is given by

$$\begin{aligned}
w(t) &= w_{\mathrm{opt}}(t) = (I_q - (E(t))'(N(t))^{-1}E(t))(D(t))'(\bar{\Sigma}(t))^{-1}(x(t) - \check{x}(t)) \\
&\quad -(E(t))'(N(t))^{-1}(C(t)(x(t) - \check{x}(t)) - \gamma^{-2}(C(t)\bar{\Sigma}(t) + (L(t))')\Xi(t)\check{x}(t)) \quad (20a)
\end{aligned}$$

and the worst case terminal state is

$$x_{\mathrm{opt}}(t_f) = (I_n - \gamma^{-2}\bar{\Sigma}(t_f)Q_f)^{-1}\check{x}(t_f) \quad (20b)$$

We have solved the H^∞-optimal control problem in a sequencial fashion. But, the second Riccati differential equation (17a) depends on solution to the first Riccati differential equation (9a), and is not the well-known control Riccati differential equation.

In the next section, we will present an explicit formula that solves Ξ as a function of $\bar{\Sigma}$ and the solution to the well-known control Riccati differential equation.

6 Finite-horizon case

The well-known control Riccati differential equation is as given below:

$$\begin{aligned}
&\dot{Z}(t) + Z(t)(A(t) - B(t)(R(t))^{-1}(P(t))') + (A(t) - B(t)(R(t))^{-1}(P(t))')'Z(t) \\
&\quad -Z(t)(B(t)(R(t))^{-1}(B(t))' - \gamma^{-2}D(t)(D(t))')Z(t) + Q(t) - P(t)(R(t))^{-1}(P(t))' \\
&\quad = \mathbf{0}_{n\times n}; \;\; Z(t_f) = Q_f \quad (21)
\end{aligned}$$

Based on the terminal condition for Ξ, we conjecture that

$$\Xi(t) = (I_n - \gamma^{-2}Z(t)\bar{\Sigma}(t))^{-1}Z(t) =: \bar{\Xi}(t) \tag{22}$$

We will show that $\bar{\Xi}$ satisfies the Riccati differential equation (17a) by brute force. Clearly, $\bar{\Xi}$ satisfies the terminal condition of (17a). To simplify things, we will introduce the notations: $\bar{A}(t) := A(t) - B(t)(R(t))^{-1}(P(t))'$, $\bar{S}(t) := B(t)(R(t))^{-1}(B(t))' - \gamma^{-2}D(t)(D(t))'$, $\bar{Q}(t) := Q(t) - P(t)(R(t))^{-1}(P(t))'$, $\tilde{A}(t) := A(t) - L(t)(N(t))^{-1}C(t)$, $\tilde{R} := (C(t))'(N(t))^{-1}C(t) - \gamma^{-2}Q(t)$, $\tilde{M} := D(t)(D(t))' - L(t)(N(t))^{-1}(L(t))'$, $\forall t \in [t_0, t_f] \subset \mathbb{R}$. Then, we have

$$\begin{aligned}
\dot{\bar{\Sigma}} &= \tilde{A}\bar{\Sigma} + \bar{\Sigma}\tilde{A}' - \bar{\Sigma}\tilde{R}\bar{\Sigma} + \tilde{M}; \quad \bar{\Sigma}(t_0) = Q_0^{-1} \\
\dot{Z} &= -\bar{A}'Z - Z\bar{A} + Z\bar{S}Z - \bar{Q}; \quad Z(t_f) = Q_f
\end{aligned}$$

and

$$\begin{aligned}
\dot{\bar{\Xi}} &= (I_n - \gamma^{-2}Z\bar{\Sigma})^{-1}\dot{Z} - (I_n - \gamma^{-2}Z\bar{\Sigma})^{-1}(-\gamma^{-2}\dot{Z}\bar{\Sigma} - \gamma^{-2}Z\dot{\bar{\Sigma}})(I_n - \gamma^{-2}Z\bar{\Sigma})^{-1}Z \\
&= (I_n - \gamma^{-2}Z\bar{\Sigma})^{-1}(-\bar{A}'Z - Z\bar{A} + Z\bar{S}Z - \bar{Q}) + \gamma^{-2}(I_n - \gamma^{-2}Z\bar{\Sigma})^{-1}\dot{Z}\bar{\Sigma}\bar{\Xi} \\
&\quad + \gamma^{-2}\bar{\Xi}\dot{\bar{\Sigma}}\bar{\Xi} \\
&= -(I_n - \gamma^{-2}Z\bar{\Sigma})^{-1}\bar{A}'Z - \bar{\Xi}\bar{A} + \bar{\Xi}\bar{S}Z - (I_n - \gamma^{-2}Z\bar{\Sigma})^{-1}\bar{Q} \\
&\quad + \gamma^{-2}(I_n - \gamma^{-2}Z\bar{\Sigma})^{-1}(-\bar{A}'Z - Z\bar{A} + Z\bar{S}Z - \bar{Q})\bar{\Sigma}\bar{\Xi} \\
&\quad + \gamma^{-2}\bar{\Xi}(\tilde{A}\bar{\Sigma} + \bar{\Sigma}\tilde{A}' - \bar{\Sigma}\tilde{R}\bar{\Sigma} + \tilde{M})\bar{\Xi} \\
&= -(I_n - \gamma^{-2}Z\bar{\Sigma})^{-1}\bar{A}'Z - \bar{\Xi}\bar{A} + \bar{\Xi}\bar{S}Z - (I_n - \gamma^{-2}Z\bar{\Sigma})^{-1}\bar{Q} \\
&\quad - \gamma^{-2}(I_n - \gamma^{-2}Z\bar{\Sigma})^{-1}\bar{A}'Z\bar{\Sigma}\bar{\Xi} - \gamma^{-2}\bar{\Xi}\bar{A}\bar{\Sigma}\bar{\Xi} + \gamma^{-2}\bar{\Xi}\bar{S}Z\bar{\Sigma}\bar{\Xi} - \gamma^{-2}(I_n \\
&\quad - \gamma^{-2}Z\bar{\Sigma})^{-1}\bar{Q}\bar{\Sigma}\bar{\Xi} + \gamma^{-2}\bar{\Xi}\tilde{A}\bar{\Sigma}\bar{\Xi} + \gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{A}'\bar{\Xi} - \gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{R}\bar{\Sigma}\bar{\Xi} + \gamma^{-2}\bar{\Xi}\tilde{M}\bar{\Xi} \\
&= -(I_n - \gamma^{-2}Z\bar{\Sigma})^{-1}\bar{A}'(I_n + \gamma^{-2}Z\bar{\Sigma}(I_n - \gamma^{-2}Z\bar{\Sigma})^{-1})Z - \bar{\Xi}\bar{A} \\
&\quad + \bar{\Xi}\bar{S}(I_n + \gamma^{-2}Z\bar{\Sigma}(I_n - \gamma^{-2}Z\bar{\Sigma})^{-1})Z - (I_n - \gamma^{-2}Z\bar{\Sigma})^{-1}\bar{Q} \\
&\quad - \gamma^{-2}\bar{\Xi}\bar{A}\bar{\Sigma}\bar{\Xi} - \gamma^{-2}(I_n - \gamma^{-2}Z\bar{\Sigma})^{-1}\bar{Q}\bar{\Sigma}\bar{\Xi} \\
&\quad + \gamma^{-2}\bar{\Xi}\tilde{A}\bar{\Sigma}\bar{\Xi} + \gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{A}'\bar{\Xi} - \gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{R}\bar{\Sigma}\bar{\Xi} + \gamma^{-2}\bar{\Xi}\tilde{M}\bar{\Xi} \\
&= -(I_n - \gamma^{-2}Z\bar{\Sigma})^{-1}\bar{A}'\bar{\Xi} - \bar{\Xi}\bar{A} + \bar{\Xi}\bar{S}\bar{\Xi} - (I_n - \gamma^{-2}Z\bar{\Sigma})^{-1}\bar{Q} \\
&\quad - \gamma^{-2}\bar{\Xi}\bar{A}\bar{\Sigma}\bar{\Xi} - \gamma^{-2}(I_n - \gamma^{-2}Z\bar{\Sigma})^{-1}\bar{Q}\bar{\Sigma}\bar{\Xi} \\
&\quad + \gamma^{-2}\bar{\Xi}\tilde{A}\bar{\Sigma}\bar{\Xi} + \gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{A}'\bar{\Xi} - \gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{R}\bar{\Sigma}\bar{\Xi} + \gamma^{-2}\bar{\Xi}\tilde{M}\bar{\Xi}
\end{aligned}$$

Note that $(\bar{\Xi}(t))' = Z(t)(I_n - \gamma^{-2}\bar{\Sigma}(t)Z(t))^{-1} = Z(t)(I_n + \gamma^{-2}\bar{\Sigma}(t)(I_n - \gamma^{-2}Z(t)\bar{\Sigma}(t))^{-1}Z(t)) = Z(t) + \gamma^{-2}Z(t)\bar{\Sigma}(t)(I_n - \gamma^{-2}Z(t)\bar{\Sigma}(t))^{-1}Z(t) = (I_n + \gamma^{-2}Z(t)\bar{\Sigma}(t)(I_n - \gamma^{-2}Z(t)\bar{\Sigma}(t))^{-1})Z(t) = (I_n - \gamma^{-2}Z(t)\bar{\Sigma}(t))^{-1}Z(t) = \bar{\Xi}(t)$, $\forall t \in [t_0, t_f] \subset \mathbb{R}$, by standard matrix identity. Then, $\bar{\Xi}(t) \in \mathcal{S}_n$. Using this in the $\dot{\bar{\Xi}}$ formula, we have

$$\begin{aligned}
\dot{\bar{\Xi}} &= -(I_n - \gamma^{-2}Z\bar{\Sigma})^{-1}\bar{A}'\bar{\Xi} - \bar{\Xi}\bar{A}(I_n + \gamma^{-2}\bar{\Sigma}\bar{\Xi}) + \bar{\Xi}\bar{S}\bar{\Xi} \\
&\quad - (I_n - \gamma^{-2}Z\bar{\Sigma})^{-1}\bar{Q}(I_n + \gamma^{-2}\bar{\Sigma}\bar{\Xi})
\end{aligned}$$

$$
\begin{aligned}
&+\gamma^{-2}\bar{\Xi}\tilde{A}\bar{\Sigma}\bar{\Xi}+\gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{A}'\bar{\Xi}-\gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{R}\bar{\Sigma}\bar{\Xi}+\gamma^{-2}\bar{\Xi}\tilde{M}\bar{\Xi}\\
= &-(I_n-\gamma^{-2}Z\bar{\Sigma})^{-1}\bar{A}'\bar{\Xi}-\bar{\Xi}\bar{A}(I_n+\gamma^{-2}\bar{\Sigma}Z(I_n-\gamma^{2}\bar{\Sigma}Z)^{-1})+\bar{\Xi}\bar{S}\bar{\Xi}\\
&-(I_n-\gamma^{-2}Z\bar{\Sigma})^{-1}\bar{Q}(I_n+\gamma^{-2}\bar{\Sigma}Z(I_n-\gamma^{2}\bar{\Sigma}Z)^{-1})\\
&+\gamma^{-2}\bar{\Xi}\tilde{A}\bar{\Sigma}\bar{\Xi}+\gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{A}'\bar{\Xi}-\gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{R}\bar{\Sigma}\bar{\Xi}+\gamma^{-2}\bar{\Xi}\tilde{M}\bar{\Xi}\\
= &-(I_n-\gamma^{-2}Z\bar{\Sigma})^{-1}\bar{A}'\bar{\Xi}-\bar{\Xi}\bar{A}(I_n-\gamma^{2}\bar{\Sigma}Z)^{-1}+\bar{\Xi}\bar{S}\bar{\Xi}\\
&-(I_n-\gamma^{-2}Z\bar{\Sigma})^{-1}\bar{Q}(I_n-\gamma^{2}\bar{\Sigma}Z)^{-1}\\
&+\gamma^{-2}\bar{\Xi}\tilde{A}\bar{\Sigma}\bar{\Xi}+\gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{A}'\bar{\Xi}-\gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{R}\bar{\Sigma}\bar{\Xi}+\gamma^{-2}\bar{\Xi}\tilde{M}\bar{\Xi}
\end{aligned}
$$

Plug the expressions for $\dot{\bar{\Xi}}$ and $\bar{\Xi}$ into the right-hand-side of (17a), we have

$$
\begin{aligned}
&\dot{\bar{\Xi}}+\bar{\Xi}(\bar{A}+\gamma^{-2}\bar{\Sigma}\bar{Q})+(\bar{A}+\gamma^{-2}\bar{\Sigma}\bar{Q})'\bar{\Xi}-\bar{\Xi}((B+\gamma^{-2}\bar{\Sigma}P)R^{-1}(B+\gamma^{-2}\bar{\Sigma}P)'\\
&\quad-\gamma^{-2}(\bar{\Sigma}C'+L)N^{-1}(\bar{\Sigma}C'+L)')\bar{\Xi}+\bar{Q}\\
&= -(I_n-\gamma^{-2}Z\bar{\Sigma})^{-1}\bar{A}'\bar{\Xi}-\bar{\Xi}\bar{A}(I_n-\gamma^{2}\bar{\Sigma}Z)^{-1}-(I_n-\gamma^{-2}Z\bar{\Sigma})^{-1}\bar{Q}(I_n-\gamma^{2}\bar{\Sigma}Z)^{-1}\\
&\quad+\gamma^{-2}\bar{\Xi}\tilde{A}\bar{\Sigma}\bar{\Xi}+\gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{A}'\bar{\Xi}-\gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{R}\bar{\Sigma}\bar{\Xi}+\gamma^{-2}\bar{\Xi}\tilde{M}\bar{\Xi}+\bar{\Xi}\bar{S}\bar{\Xi}\\
&\quad+\bar{\Xi}(\bar{A}+\gamma^{-2}\bar{\Sigma}\bar{Q})+(\bar{A}+\gamma^{-2}\bar{\Sigma}\bar{Q})'\bar{\Xi}-\bar{\Xi}((B+\gamma^{-2}\bar{\Sigma}P)R^{-1}(B+\gamma^{-2}\bar{\Sigma}P)'\\
&\quad-\gamma^{-2}(\bar{\Sigma}C'+L)N^{-1}(\bar{\Sigma}C'+L)')\bar{\Xi}+\bar{Q}\\
&= -(I_n-\gamma^{-2}Z\bar{\Sigma})^{-1}\bar{A}'\bar{\Xi}+\bar{A}'\bar{\Xi}-\bar{\Xi}\bar{A}(I_n-\gamma^{2}\bar{\Sigma}Z)^{-1}+\bar{\Xi}\bar{A}\\
&\quad+\gamma^{-2}\bar{\Xi}\tilde{A}\bar{\Sigma}\bar{\Xi}+\gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{A}'\bar{\Xi}-\gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{R}\bar{\Sigma}\bar{\Xi}+\gamma^{-2}\bar{\Xi}\tilde{M}\bar{\Xi}+\bar{\Xi}\bar{S}\bar{\Xi}\\
&\quad-\bar{\Xi}((B+\gamma^{-2}\bar{\Sigma}P)R^{-1}(B+\gamma^{-2}\bar{\Sigma}P)'-\gamma^{-2}(\bar{\Sigma}C'+L)N^{-1}(\bar{\Sigma}C'+L)')\bar{\Xi}\\
&\quad+\bar{Q}-(I_n-\gamma^{-2}Z\bar{\Sigma})^{-1}\bar{Q}(I_n-\gamma^{2}\bar{\Sigma}Z)^{-1}+\bar{\Xi}\gamma^{-2}\bar{\Sigma}\bar{Q}+\bar{Q}\gamma^{-2}\bar{\Sigma}\bar{\Xi}\\
&= (I_n-\gamma^{-2}Z\bar{\Sigma})^{-1}(I_n-\gamma^{-2}Z\bar{\Sigma}-I_n)\bar{A}'\bar{\Xi}+\bar{\Xi}\bar{A}(I_n-\gamma^{-2}\bar{\Sigma}Z-I_n)(I_n-\gamma^{2}\bar{\Sigma}Z)^{-1}\\
&\quad+\gamma^{-2}\bar{\Xi}\tilde{A}\bar{\Sigma}\bar{\Xi}+\gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{A}'\bar{\Xi}-\gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{R}\bar{\Sigma}\bar{\Xi}+\bar{\Xi}(\bar{S}+\gamma^{-2}\tilde{M})\bar{\Xi}\\
&\quad-\bar{\Xi}(BR^{-1}B'+\gamma^{-2}\bar{\Sigma}PR^{-1}B'+BR^{-1}P'\gamma^{-2}\bar{\Sigma}+\gamma^{-2}\bar{\Sigma}PR^{-1}P'\gamma^{-2}\bar{\Sigma}\\
&\quad-\gamma^{-2}\bar{\Sigma}C'N^{-1}C\bar{\Sigma}-\gamma^{-2}LN^{-1}C\bar{\Sigma}-\gamma^{-2}\bar{\Sigma}C'N^{-1}L'-LN^{-1}\gamma^{-2}L')\bar{\Xi}\\
&\quad+\bar{Q}(I_n+\gamma^{-2}\bar{\Sigma}Z(I_n-\gamma^{2}\bar{\Sigma}Z)^{-1})-(I_n-\gamma^{-2}Z\bar{\Sigma})^{-1}\bar{Q}(I_n-\gamma^{2}\bar{\Sigma}Z)^{-1}+\bar{\Xi}\gamma^{-2}\bar{\Sigma}\bar{Q}\\
&= -\gamma^{-2}\bar{\Xi}\bar{\Sigma}\bar{A}'\bar{\Xi}-\gamma^{-2}\bar{\Xi}\bar{A}\bar{\Sigma}\bar{\Xi}+\gamma^{-2}\bar{\Xi}\tilde{A}\bar{\Sigma}\bar{\Xi}+\gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{A}'\bar{\Xi}-\gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{R}\bar{\Sigma}\bar{\Xi}\\
&\quad+\bar{\Xi}(BR^{-1}B'-\gamma^{-2}LN^{-1}L')\bar{\Xi}\\
&\quad-\bar{\Xi}(BR^{-1}B'+\gamma^{-2}\bar{\Sigma}PR^{-1}B'+BR^{-1}P'\gamma^{-2}\bar{\Sigma}+\gamma^{-2}\bar{\Sigma}PR^{-1}P'\gamma^{-2}\bar{\Sigma}\\
&\quad-\gamma^{-2}\bar{\Sigma}C'N^{-1}C\bar{\Sigma}-\gamma^{-2}LN^{-1}C\bar{\Sigma}-\gamma^{-2}\bar{\Sigma}C'N^{-1}L'-\gamma^{-2}LN^{-1}L')\bar{\Xi}\\
&\quad+\bar{Q}(I_n-\gamma^{2}\bar{\Sigma}Z)^{-1}-(I_n-\gamma^{-2}Z\bar{\Sigma})^{-1}\bar{Q}(I_n-\gamma^{2}\bar{\Sigma}Z)^{-1}+\bar{\Xi}\gamma^{-2}\bar{\Sigma}\bar{Q}\\
&= \gamma^{-2}\bar{\Xi}(\tilde{A}-\bar{A})\bar{\Sigma}\bar{\Xi}+\gamma^{-2}\bar{\Xi}\bar{\Sigma}(\tilde{A}'-\bar{A}')\bar{\Xi}-\gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{R}\bar{\Sigma}\bar{\Xi}\\
&\quad-\bar{\Xi}(\gamma^{-2}\bar{\Sigma}PR^{-1}B'+BR^{-1}P'\gamma^{-2}\bar{\Sigma}+\gamma^{-2}\bar{\Sigma}PR^{-1}P'\gamma^{-2}\bar{\Sigma}\\
&\quad-\gamma^{-2}\bar{\Sigma}C'N^{-1}C\bar{\Sigma}-\gamma^{-2}LN^{-1}C\bar{\Sigma}-\gamma^{-2}\bar{\Sigma}C'N^{-1}L')\bar{\Xi}\\
&\quad+(I_n-\gamma^{-2}Z\bar{\Sigma})^{-1}(I_n-\gamma^{-2}Z\bar{\Sigma}-I_n)\bar{Q}(I_n-\gamma^{2}\bar{\Sigma}Z)^{-1}+\gamma^{-2}\bar{\Xi}\bar{\Sigma}\bar{Q}\\
&= \gamma^{-2}\bar{\Xi}(-LN^{-1}C+BR^{-1}P')\bar{\Sigma}\bar{\Xi}+\gamma^{-2}\bar{\Xi}\bar{\Sigma}(-LN^{-1}C+BR^{-1}P')'\bar{\Xi}-\gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{R}\bar{\Sigma}\bar{\Xi}\\
&\quad-\bar{\Xi}(\gamma^{-2}\bar{\Sigma}PR^{-1}B'+BR^{-1}P'\gamma^{-2}\bar{\Sigma}+\gamma^{-2}\bar{\Sigma}PR^{-1}P'\gamma^{-2}\bar{\Sigma}\\
&\quad-\gamma^{-2}\bar{\Sigma}C'N^{-1}C\bar{\Sigma}-\gamma^{-2}LN^{-1}C\bar{\Sigma}-\gamma^{-2}\bar{\Sigma}C'N^{-1}L')\bar{\Xi}
\end{aligned}
$$

$$
\begin{aligned}
&-\gamma^{-2}\bar{\Xi}\bar{\Sigma}\bar{Q}(I_n-\gamma^2\bar{\Sigma}Z)^{-1}+\gamma^{-2}\bar{\Xi}\bar{\Sigma}\bar{Q} \\
=&-\gamma^{-2}\bar{\Xi}\bar{\Sigma}\tilde{R}\bar{\Sigma}\bar{\Xi}-\gamma^{-2}\bar{\Xi}(\gamma^{-2}\bar{\Sigma}PR^{-1}P'\bar{\Sigma}-\bar{\Sigma}C'N^{-1}C\bar{\Sigma})\bar{\Xi} \\
&+\gamma^{-2}\bar{\Xi}\bar{\Sigma}\bar{Q}(I_n-\gamma^2\bar{\Sigma}Z-I_n)(I_n-\gamma^2\bar{\Sigma}Z)^{-1} \\
=&\ \gamma^{-2}\bar{\Xi}\bar{\Sigma}(-\tilde{R}-\gamma^{-2}PR^{-1}P'+C'N^{-1}C-\gamma^{-2}\bar{Q})\bar{\Sigma}\bar{\Xi} \\
=&\ \gamma^{-2}\bar{\Xi}\bar{\Sigma}(-C'N^{-1}C+\gamma^{-2}Q-\gamma^{-2}PR^{-1}P'+C'N^{-1}C-\gamma^{-2}\bar{Q})\bar{\Sigma}\bar{\Xi}=\mathbf{0}_{n\times n}
\end{aligned}
$$

Hence, $\bar{\Xi}$ is the solution to the Riccati differential equation (17a) by uniqueness of solutions to ordinary differential equations.

Thus, under the assumption

Assumption 3. *The generalized Riccati differential equations (9a) and (21) admits solutions on the interval $[t_0, t_f]$, and they satisfy the spectral radius condition: all eigenvalues of the matrix $\bar{\Sigma}(t)Z(t)$ (which are nonnegative real to begin with) are less than γ^2, $\forall t \in [t_0, t_f] \subset \mathbb{R}$.*

Then, the upper value of the game with cost function $J_\gamma(\mu, \nu, \check{x}_0, Q_0, Q_f)$ (5) is finite, and is equal to $|\check{x}_0|^2_{\Xi(t_0)} = |\check{x}_0|^2_{(I_n-\gamma^{-2}Z(t_0)Q_0^{-1})^{-1}Z(t_0)}$.

We summarize the above finding in the following theorem.

Theorem 1. *Consider the continuous-time linear time-varying system (1) with the soft-constrained zero-sum game cost function (5) with $Q_0 \in \mathcal{S}_{+\,n}$ and $Q_f \in \mathcal{S}_{\text{psd}\,n}$. Let the matrix-valued functions A, B, D, C, E, H, and G be bounded and $\mathcal{B}_{\text{B}}(\mathbb{R})$-measurable on the interval $[t_0, t_f]$. Fix any $\gamma \in \mathbb{R}_+$, if Assumptions 1 — 3 hold, then $\bar{\Sigma}(t) \in \mathcal{S}_{+\,n}$, $Z(t) \in \mathcal{S}_{\text{psd}\,n}$, $\forall t \in [t_0, t_f] \subset \mathbb{R}$, and the upper value of the game is finite and given by $|\check{x}_0|^2_{\Xi(t_0)} = |\check{x}_0|^2_{(I_n-\gamma^{-2}Z(t_0)Q_0^{-1})^{-1}Z(t_0)}$. Furthermore, a controller that achieves the upper value is given by*

$$
\begin{aligned}
\dot{\check{x}}(t) &= (A(t)+\gamma^{-2}\bar{\Sigma}(t)Q(t))\check{x}(t)+B(t)u(t)+\gamma^{-2}\bar{\Sigma}(t)P(t)u(t)+(\bar{\Sigma}(t)(C(t))' \\
&\quad +L(t))(N(t))^{-1}(y(t)-C(t)\check{x}(t)); \quad \check{x}(t_0)=\check{x}_0 \qquad (23a)\\
u(t) &= \mu_{\text{opt}}(t, \check{x}_0, y_{[t_0,t]}) \\
&= -(R(t))^{-1}((B(t))'Z(t)+(P(t))')(I_n-\gamma^{-2}\bar{\Sigma}(t)Z(t))^{-1}\check{x}(t) \qquad (23b)
\end{aligned}
$$

Define

$$
\hat{x}(t) := (I_n-\gamma^{-2}\bar{\Sigma}(t)Z(t))^{-1}\check{x}(t) \qquad (24)
$$

Then, the corresponding worst-case disturbance waveform and worst-case initial condition is given by

$$
\begin{aligned}
\dot{\hat{x}}(t) &= (A(t)-B(t)(R(t))^{-1}((B(t))'Z(t)+(P(t))')+\gamma^{-2}D(t)(D(t))'Z(t))\hat{x}(t) \qquad (25a)\\
w(t) &= w_{\text{opt}}(t)=\gamma^{-2}(D(t))'Z(t)\hat{x}(t)+(E(t))'(N(t))^{-1}(C(t)\bar{\Sigma}(t)+(L(t))')(\bar{\Sigma}(t))^{-1} \\
&\quad \cdot(\hat{x}(t)-x(t))+(D(t))'(\bar{\Sigma}(t))^{-1}(x(t)-\hat{x}(t)) \qquad (25b)\\
x_0 &= \hat{x}(t_0)=(I_n-\gamma^{-2}Q_0^{-1}Z(t_0))^{-1}\check{x}_0 \qquad (25c)
\end{aligned}
$$

Proof. By Proposition 9.4 of Başar and Bernhard (1995) and the fact that $\bar{\Sigma}$ and Z are solution to the generalized Riccati differential equations (9a) and (21), respectively, we have $\bar{\Sigma}(t), Z(t) \in \mathcal{S}_{\text{psd}\,n}$, $\forall t \in [t_0, t_f] \subset \mathbb{R}$. Since $Q_0 \in \mathcal{S}_{+\,n}$, then $\bar{\Sigma}(t) \in \mathcal{S}_{+\,n}$, $\forall t \in [t_0, t_f] \subset \mathbb{R}$.

Note that (17b) is identical to (23b). Note also that the worst-case disturbance policy (20a) is identical to (25b) as follows.

$$\begin{aligned}
w_{\text{opt}}(t) &= E'N^{-1}(C\check{x} + \gamma^{-2}(C\bar{\Sigma} + L')\Xi\check{x} - Cx) + (I_q - E'N^{-1}E)D'\bar{\Sigma}^{-1}(x - \check{x}) \\
&= E'N^{-1}C(\hat{x} - x) + \gamma^{-2}E'N^{-1}L'Z\hat{x} \\
&\quad + (I_q - E'N^{-1}E)D'\bar{\Sigma}^{-1}(x - \hat{x}) + \gamma^{-2}(I_q - E'N^{-1}E)D'Z\hat{x} \\
&= E'N^{-1}C(\hat{x} - x) + (I_q - E'N^{-1}E)D'\bar{\Sigma}^{-1}(x - \hat{x}) + \gamma^{-2}D'Z\hat{x} \\
&= \gamma^{-2}D'Z\hat{x} + E'N^{-1}(C\bar{\Sigma} + L')\bar{\Sigma}^{-1}(\hat{x} - x) + D'\bar{\Sigma}^{-1}(x - \hat{x})
\end{aligned}$$

We have the following equivalent form of the cost function (5), by (19),

$$\begin{aligned}
&J_\gamma(\mu, \nu, \check{x}_0, Q_0, Q_f) = -\left| x(t_f) - (I_n - \gamma^{-2}\bar{\Sigma}(t_f)Q_f)^{-1}\check{x}(t_f) \right|^2_{\gamma^2(\bar{\Sigma}(t_f))^{-1} - Q_f} \\
&+ |\check{x}_0|^2_{\Xi(t_0)} + \int_{t_0}^{t_f} (\left| u(t) + (R(t))^{-1}((B(t))'Z(t) + (P(t))')\hat{x}(t) \right|^2_{R(t)} \\
&-\gamma^2 \left| w(t) - \gamma^{-2}(D(t))'Z(t)\hat{x}(t) - (E(t))'(N(t))^{-1}(C(t)\bar{\Sigma}(t) + (L(t))')(\bar{\Sigma}(t))^{-1} \right. \\
&\left. \cdot(\hat{x}(t) - x(t)) - (D(t))'(\bar{\Sigma}(t))^{-1}(x(t) - \hat{x}(t)) \right|^2) \, \mathrm{d}t \qquad (26)
\end{aligned}$$

Hence, the controller μ_{opt} given by (23) achieves the upper value $|\check{x}_0|^2_{\Xi(t_0)}$.

To show the corresponding worst-case disturbance waveform and worst-case initial condition is given by (25), we will obtain the dynamics for $\hat{x}$ under the optimal control law (23) and the maximizer policy (20).

$$\begin{aligned}
\dot{\hat{x}} &= (I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}\dot{\check{x}} + \gamma^{-2}(I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}(\dot{\bar{\Sigma}}Z + \bar{\Sigma}\dot{Z})(I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}\check{x} \\
&= (I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}((A + \gamma^{-2}\bar{\Sigma}Q)\check{x} - (B + \gamma^{-2}\bar{\Sigma}P)R^{-1}(B'Z + P')\hat{x} + \gamma^{-2}(\bar{\Sigma}C' \\
&\quad + L)N^{-1}(C\bar{\Sigma} + L')\Xi\check{x}) + \gamma^{-2}(I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}\dot{\bar{\Sigma}}Z\hat{x} + \gamma^{-2}(I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}\dot{Z}\hat{x} \\
&= (I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}(A + \gamma^{-2}\bar{\Sigma}Q)(I_n - \gamma^{-2}\bar{\Sigma}Z)\hat{x} \\
&\quad - (I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}(B + \gamma^{-2}\bar{\Sigma}P)R^{-1}(B'Z + P')\hat{x} \\
&\quad + \gamma^{-2}(I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}(\bar{\Sigma}C' + L)N^{-1}(C\bar{\Sigma} + L')Z\hat{x} \\
&\quad + \gamma^{-2}(I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}\tilde{A}\bar{\Sigma}Z\hat{x} + \gamma^{-2}(I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}\tilde{A}'Z\hat{x} \\
&\quad - \gamma^{-2}(I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}\tilde{R}\bar{\Sigma}Z\hat{x} + \gamma^{-2}(I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}\tilde{M}Z\hat{x} \\
&\quad - \gamma^{-2}(I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}\bar{A}'Z\hat{x} - \gamma^{-2}(I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}Z\bar{A}\hat{x} \\
&\quad + \gamma^{-2}(I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}Z\bar{S}Z\hat{x} - \gamma^{-2}(I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}\bar{Q}\hat{x} \\
&= (I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}(A(I_n - \gamma^{-2}\bar{\Sigma}Z) + \gamma^{-2}\tilde{A}\bar{\Sigma}Z)\hat{x} \\
&\quad + \gamma^{-2}(I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}Q(I_n - \gamma^{-2}\bar{\Sigma}Z)\hat{x} \\
&\quad - (I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}(B + \gamma^{-2}\bar{\Sigma}P)R^{-1}(B'Z + P')\hat{x} \\
&\quad + \gamma^{-2}(I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}(\bar{\Sigma}C' + L)N^{-1}(C\bar{\Sigma} + L')Z\hat{x} \\
&\quad + \gamma^{-2}(I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}(\tilde{A}' - \bar{A}')Z\hat{x} - \gamma^{-2}(I_n - \gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}Z\bar{A}\hat{x}
\end{aligned}$$

$$
\begin{aligned}
&-\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}\tilde{R}\bar{\Sigma}Z\hat{x}+\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\tilde{M}Z\hat{x}\\
&+\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}Z\bar{S}Z\hat{x}-\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}\bar{Q}\hat{x}\\
=\ &(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}(A-\gamma^{-2}LN^{-1}C\bar{\Sigma}Z)\hat{x}\\
&+\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}(Q(I_n-\gamma^{-2}\bar{\Sigma}Z)-\bar{Q})\hat{x}\\
&-(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}(BR^{-1}B'Z+\gamma^{-2}\bar{\Sigma}PR^{-1}B'Z+BR^{-1}P'+\gamma^{-2}\bar{\Sigma}PR^{-1}P')\hat{x}\\
&+\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}(\bar{\Sigma}C'N^{-1}C\bar{\Sigma}+LN^{-1}C\bar{\Sigma}+\bar{\Sigma}C'N^{-1}L'+LN^{-1}L')Z\hat{x}\\
&+\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}(PR^{-1}B'-C'N^{-1}L')Z\hat{x}-\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}Z\bar{A}\hat{x}\\
&-\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}\tilde{R}\bar{\Sigma}Z\hat{x}+\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\tilde{M}Z\hat{x}\\
&+\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}Z\bar{S}Z\hat{x}\\
=\ &(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}A\hat{x}-\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}Z\bar{A}\hat{x}\\
&+\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}(PR^{-1}P'-\gamma^{-2}Q\bar{\Sigma}Z)\hat{x}\\
&-(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}(BR^{-1}B'Z+BR^{-1}P'+\gamma^{-2}\bar{\Sigma}PR^{-1}P')\hat{x}\\
&+\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}(\bar{\Sigma}C'N^{-1}C\bar{\Sigma}+LN^{-1}L')Z\hat{x}\\
&-\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}(C'N^{-1}C-\gamma^{-2}Q)\bar{\Sigma}Z\hat{x}+\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\tilde{M}Z\hat{x}\\
&+\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}Z\bar{S}Z\hat{x}\\
=\ &A\hat{x}+\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}ZBR^{-1}P'\hat{x}\\
&-(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}(BR^{-1}B'Z+BR^{-1}P')\hat{x}\\
&+\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}(LN^{-1}L')Z\hat{x}\\
&+\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\tilde{M}Z\hat{x}\\
&+\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}Z\bar{S}Z\hat{x}\\
=\ &A\hat{x}+\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}ZBR^{-1}P'\hat{x}-(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}BR^{-1}P'\hat{x}\\
&-(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}BR^{-1}B'Z\hat{x}+\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}Z\bar{S}Z\hat{x}\\
&+\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}DD'Z\hat{x}\\
=\ &A\hat{x}-BR^{-1}P'\hat{x}\\
&-(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{S}Z\hat{x}+\gamma^{-2}(I_n-\gamma^{-2}\bar{\Sigma}Z)^{-1}\bar{\Sigma}Z\bar{S}Z\hat{x}\\
=\ &A\hat{x}-BR^{-1}P'\hat{x}-BR^{-1}B'Z\hat{x}+\gamma^{-2}DD'Z\hat{x}\\
=\ &A\hat{x}-BR^{-1}(B'Z+P')\hat{x}+\gamma^{-2}DD'Z\hat{x}
\end{aligned}
$$

The dynamics of x under the optimal control law (23) and worst-case disturbance policy (20) is given by

$$\dot{x} = Ax-BR^{-1}(B'Z+P')\hat{x}+\gamma^{-2}DD'Z\hat{x}+D(D'-EN^{-1}(L'+C\bar{\Sigma}))\bar{\Sigma}^{-1}(x-\hat{x})$$

Since the worst choice for $x(t_f)$ is $\hat{x}(t_f)$, then $x(t)\equiv\hat{x}(t)$, $\forall t\in[t_0,t_f]\subset\mathbb{R}$ by the uniqueness of solutions to ordinary differential equations. Hence, $w(t)\equiv\gamma^{-2}(D(t))'Z(t)\hat{x}(t)$ since $x\equiv\hat{x}$ on the interval $[t_0,t_f]$. This proves that the worst-case choice for x_0 is $\hat{x}(t_0)$ and the worst-case disturbance waveform is given by (25b).

This completes the proof of the theorem. □

7 Infinite-horizon case

In this section, we consider the infinite-horizon case of the H^∞-optimal control problem under imperfect state measurements. The system under consideration is (1), where A, B, D, C, E, H, and G are constant matrices of appropriate dimensions. The time interval under consideration is $\mathbb{R}$, and $x(-\infty) = \check{x}_0 = \mathbf{0}_n$, $w_{(-\infty,\infty)} \in \bar{\mathrm{L}}_2(\mathbb{R}, \mathbb{R}^q)$, $Q_0 = \infty I_n$, and $Q_f = \mathbf{0}_{n\times n}$. The cost function is

$$J_{\gamma\infty}(\mu,\nu) = \lim_{t_0\downarrow-\infty, t_f\uparrow\infty} J_\gamma(\mu,\nu,\check{x}(t_0),\infty I_n, \mathbf{0}_{n\times n}) \tag{27}$$

where μ is restricted to internally stabilizing causal linear time-invariant controllers, i. e., we have the set

$$\begin{aligned}\mathcal{M} := \big\{ \mu : \mathbb{R}\times\mathbb{R}^n\times\bar{\mathrm{L}}_2(\mathbb{R},\mathbb{R}^m) \to \mathbb{R}^p \;\big|\; &\mu \text{ is an internally stabilizing} \\ &\text{linear time-invariant causal controller.} \big\}\end{aligned}$$

Then, the generalized Riccati differential equations (9a) and (21) are given by

$$\dot{\bar{\Sigma}} = \tilde{A}\bar{\Sigma} + \bar{\Sigma}\tilde{A}' - \bar{\Sigma}\tilde{R}\bar{\Sigma} + \tilde{M}; \quad \bar{\Sigma}(t_0) = \mathbf{0}_{n\times n} \tag{28}$$

$$\dot{Z} = -\bar{A}'Z - Z\bar{A} + Z\bar{S}Z - \bar{Q}; \quad Z(t_f) = \mathbf{0}_{n\times n} \tag{29}$$

where $t_0 \downarrow -\infty$ and $t_f \uparrow \infty$. We will denote the solution to the above generalized Riccati differential equations by $\bar{\Sigma}(t;t_0)$ and $Z(t;t_f)$. We seek stabilizing solutions to the generalized algebraic Riccati equations

$$\tilde{A}\bar{\Sigma}_\infty + \bar{\Sigma}_\infty\tilde{A}' - \bar{\Sigma}_\infty\tilde{R}\bar{\Sigma}_\infty + \tilde{M} = \mathbf{0}_{n\times n} \tag{30}$$

$$\bar{A}'Z_\infty + Z_\infty\bar{A} - Z_\infty\bar{S}Z_\infty + \bar{Q} = \mathbf{0}_{n\times n} \tag{31}$$

such that $\tilde{A}_f := \tilde{A} - \bar{\Sigma}_\infty\tilde{R}$, $\tilde{A}_{f1} := \tilde{A} - \bar{\Sigma}_\infty C'N^{-1}C$, $\bar{A}_{f1} := \bar{A} - BR^{-1}B'Z_\infty$, and $\bar{A}_f := \bar{A} - \bar{S}Z_\infty$ are Hurwitz matrices. Let $H_c := \begin{bmatrix} \bar{A} & -\bar{S} \\ -\bar{Q} & -\bar{A}' \end{bmatrix}$ be the Hamiltonian matrix for the control generalized algebraic Riccati equation (31) and $H_o := \begin{bmatrix} \tilde{A}' & -\tilde{R} \\ -\tilde{M} & -\tilde{A} \end{bmatrix}$ be the Hamiltonian matrix for the estimation generalized algebraic Riccati equation (30).

Then, we have the following results.

Proposition 2. *Consider the Hamiltonian $H_c \in \mathbb{R}^{2n\times 2n}$ under Assumption 2. The generalized algebraic Riccati equation (31) admits a solution $Z_\infty \in \mathbb{R}^{n\times n}$ such that the matrix $\bar{A}_f$ is a Hurwitz matrix (then Z_∞ is called the stabilizing solution) if, and only if, H_c does not have any eigenvalue on the imaginary axis and its stable invariant subspace is* $\operatorname{span}\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right)$ *with $X_1, X_2 \in \mathbb{R}^{n\times n}$ and X_1 is invertible. Furthermore, under the above condition, the stabilizing solution Z_∞ is unique and $Z_\infty = X_2X_1^{-1} \in \mathcal{S}_n$.*

Proof. This is a direct consequence of Theorem 13.5 of Zhou et al. (1996). □

Proposition 3. *Consider the Hamiltonian $H_c \in \mathbb{R}^{2n\times 2n}$ under Assumption 2. Let the generalized algebraic Riccati equation (31) admit the stabilizing solution $Z_\infty \in \mathcal{S}_n$. Then, the following statements hold.*

- *(i) Then, the pair $(\bar{A} - BR^{-1}B'Z_\infty, D)$ is stabilizable, the pair $(\bar{A} - BR^{-1}B'Z_\infty, \gamma^{-1}D'Z_\infty)$ is detectable, and therefore the pair $(\bar{A} - BR^{-1}B'Z_\infty, \gamma^{-2}Z_\infty DD'Z_\infty)$ is detectable.*
- *(ii) $Z_\infty \in \mathcal{S}_{\mathrm{psd}\,n}$ if, and only if, the matrix $\bar{A}_{f1} = \bar{A} - BR^{-1}B'Z_\infty$ is Hurwitz. Under this assumption, then the pair (A, B) is stabilizable, and the pair $(\bar{A}, B'Z_\infty)$ is detectable, and therefore the pair $(\bar{A}, Z_\infty BR^{-1}B'Z_\infty)$ is detectable, and the pair $(\bar{A}, \bar{Q} + \gamma^{-2}Z_\infty DD'Z_\infty)$ has no unobservable mode on the imaginary axis.*
- *(iii) If the pair $(\bar{A}, \bar{Q})$ is detectable and $Z_\infty \in \mathcal{S}_{\mathrm{psd}\,n}$, then, we have $\lim_{t_f\uparrow\infty} Z(t;t_f) \uparrow Z_\infty$, i. e., Z_∞ is the least positive semi-definite solution to the generalized algebraic Riccati equation (31).*
- *(iv) If the pair $(\bar{A}, \bar{Q})$ has no unobservable mode in $\overline{\mathbb{C}_-}$ and $Z_\infty \in \mathcal{S}_{\mathrm{psd}\,n}$, then we have $Z_\infty \in \mathcal{S}_{+\,n}$.*
- *(v) $Z_\infty \in \mathcal{S}_{\mathrm{psd}\,n}$ implies that the pair $(\bar{A}, \bar{Q})$ has no unobservable mode on the imaginary axis.*

Proof. (i) Since Z_∞ is the stabilizing solution to generalized algebraic Riccati equation (31), then the matrix $\bar{A} - \bar{S}Z_\infty = \bar{A}_{f1} + \gamma^{-2}DD'Z_\infty$ is Hurwitz. Then, we have the pair $(\bar{A}_{f1}, D)$ is stabilizable (with the stabilizing controller gain of $\gamma^{-2}D'Z_\infty$), and the pair $(\bar{A}_{f1}, \gamma^{-1}D'Z_\infty)$ is detectable (with the stabilizing observer gain $\gamma^{-1}D$). Then, the pair $(\bar{A}_{f1}, \gamma^{-2}Z_\infty DD'Z_\infty)$ is detectable.

(ii) Let $Z_\infty \in \mathcal{S}_{\mathrm{psd}\,n}$. Then, we rewrite the generalized algebraic Riccati equation (31) as

$$\bar{A}'_{f1}Z_\infty + Z_\infty \bar{A}_{f1} + Z_\infty BR^{-1}B'Z_\infty + \gamma^{-2}Z_\infty DD'Z_\infty + \bar{Q} = \mathbf{0}_{n\times n}$$

By the detectability of the pair $(\bar{A}_{f1}, \gamma^{-2}Z_\infty DD'Z_\infty)$, we have the pair $(\bar{A}_{f1}, Z_\infty BR^{-1}B'Z_\infty + \gamma^{-2}Z_\infty DD'Z_\infty + \bar{Q})$ is detectable (since the matrices $\bar{Q} \in \mathcal{S}_{\mathrm{psd}\,n}$ and $Z_\infty BR^{-1}B'Z_\infty \in \mathcal{S}_{\mathrm{psd}\,n}$). Then, by $Z_\infty \in \mathcal{S}_{\mathrm{psd}\,n}$, we must have the matrix $\bar{A}_{f1}$ is Hurwitz.

On the other hand, let the matrix $\bar{A}_{f1}$ be Hurwitz. Then, the algebraic Riccati equation

$$\bar{A}'\hat{Z} + \hat{Z}\bar{A} - \hat{Z}BR^{-1}B'\hat{Z} + \gamma^{-2}Z_\infty DD'Z_\infty + \bar{Q} = \mathbf{0}_{n\times n} \tag{32}$$

admits the stabilizing solution $\hat{Z} = Z_\infty$. By Theorem 13.7 of Zhou et al. (1996), we have $Z_\infty \in \mathcal{S}_{\mathrm{psd}\,n}$.

Under this assumption, by Theorem 13.7 of Zhou et al. (1996), then the pair $(\bar{A}, B)$ is stabilizable and the pair $(\bar{A}, \bar{Q} + \gamma^{-2}Z_\infty DD'Z_\infty)$ has no unobservable mode on the imaginary axis. It is clear that the pair $(\bar{A}, B)$ is stabilizable if, and only if, the pair (A, B) is stabilizable. Furthermore, the pair $(\bar{A}, B'Z_\infty)$ is detectable (with the stabilizing observer gain $-BR^{-1}$). Then, the pair $(\bar{A}, Z_\infty BR^{-1}B'Z_\infty)$ is detectable.

(iii) Let the pair $(\bar{A}, \bar{Q})$ is detectable and $Z_\infty \in \mathcal{S}_{\mathrm{psd}\,n}$. Then, by (ii), the pair (A, B) is stabilizable and $\bar{A}_{f1}$ is Hurwitz.

It should be clear that $Z(t;t_f)$ is nondecreasing in t_f, that is $Z(t;t_{f1}) \le Z(t;t_{f2})$, $\forall t, t_{f1}, t_{f2} \in \mathbb{R}$ with $t \le t_{f1} \le t_{f2}$. Then, $Z(t;t_f) \ge \mathbf{0}_{n\times n}$, $\forall t, t_f \in \mathbb{R}$ with $t \le t_f$. Since

Z_∞ is a solution to generalized Riccati differential equation (29), with terminal value $Z(t_f;t_f) = Z_\infty \geq \mathbf{0}_{n\times n}$, then, we have $Z(t;t_f) \leq Z_\infty$, $\forall t, t_f \in \mathbb{R}$ with $t \leq t_f$. Thus, $\lim_{t_f\uparrow\infty} Z(t;t_f) =: \bar{Z}_\infty$, where $\bar{Z}_\infty$ must be the least positive semi-definite solution to the generalized algebraic Riccati equation (31). This implies that $\bar{Z}_\infty \leq Z_\infty$.

Since the pair $(\bar{A}, \bar{Q})$ is detectable, then the pair $(\hat{\bar{A}}_{f1}, \bar{Q} + \bar{Z}_\infty BR^{-1}B'\bar{Z}_\infty)$ is detectable, where $\hat{\bar{A}}_{f1} := \bar{A} - BR^{-1}B'\bar{Z}_\infty$. This further implies that the pair $(\hat{\bar{A}}_{f1}, \bar{Q} + \bar{Z}_\infty BR^{-1}B'\bar{Z}_\infty + \gamma^{-2}\bar{Z}_\infty DD'\bar{Z}_\infty)$ is detectable. By

$$\hat{\bar{A}}'_{f1}\bar{Z}_\infty + \bar{Z}_\infty\hat{\bar{A}}_{f1} + \bar{Q} + \bar{Z}_\infty BR^{-1}B'\bar{Z}_\infty + \gamma^{-2}\bar{Z}_\infty DD'\bar{Z}_\infty = \mathbf{0}_{n\times n}$$

and $\bar{Z}_\infty \in \mathcal{S}_{\mathrm{psd}\,n}$, then the matrix $\hat{\bar{A}}_{f1}$ is Hurwitz.

We will show that the matrix $\bar{A} - \bar{S}\bar{Z}_\infty$ is Hurwitz, then by Proposition 2, we have $\lim_{t_f\uparrow\infty} Z(t;t_f) = \bar{Z}_\infty = Z_\infty$. This then completes the proof of (iii).

By the detectability of the pair $(\bar{A}, \bar{Q})$, there exists a $T \in \mathbb{R}^{n\times n}$ and T is invertible such that $T^{-1}\bar{A}T = \begin{bmatrix} \bar{A}_o & \mathbf{0} \\ \bar{A}_{\bar{o}o} & \bar{A}_{\bar{o}} \end{bmatrix}$, $T'\bar{Q}T = \begin{bmatrix} \bar{Q}_o & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $T^{-1}B = \begin{bmatrix} B_o \\ B_{\bar{o}} \end{bmatrix}$, $T^{-1}D = \begin{bmatrix} D_o \\ D_{\bar{o}} \end{bmatrix}$, $T^{-1}\bar{S}T^{-\prime} = \begin{bmatrix} B_oR^{-1}B'_o - \gamma^{-2}D_oD'_o & B_oR^{-1}B'_{\bar{o}} - \gamma^{-2}D_oD'_{\bar{o}} \\ B_{\bar{o}}R^{-1}B'_o - \gamma^{-2}D_{\bar{o}}D'_o & B_{\bar{o}}R^{-1}B'_{\bar{o}} - \gamma^{-2}D_{\bar{o}}D'_{\bar{o}} \end{bmatrix} =: \begin{bmatrix} \bar{S}_o & \bar{S}_{o\bar{o}} \\ \bar{S}'_{o\bar{o}} & \bar{S}_{\bar{o}} \end{bmatrix}$, the pair $(\bar{A}_o, \bar{Q}_o)$ is observable, and $\bar{A}_{\bar{o}}$ is Hurwitz. Here, we just partitioned the system into observable and unobservable parts, and all partitioning are compatible. Let $\check{\bar{Z}}_\infty := T'\bar{Z}_\infty T$ and $\check{Z} := T'ZT$, then

$$\dot{\check{Z}} + (T^{-1}\bar{A}T)'\check{Z} + \check{Z}(T^{-1}\bar{A}T) - \check{Z}(T^{-1}\bar{S}T^{-\prime})\check{Z} + T'\bar{Q}T = \mathbf{0}_{n\times n};\;\; \check{Z}(t_f) = \mathbf{0}_{n\times n} \quad (33)$$

Now, partition $\check{Z}$ compatibly as $\begin{bmatrix} \check{Z}_o & \check{Z}_{o\bar{o}} \\ \check{Z}'_{o\bar{o}} & \check{Z}_{\bar{o}} \end{bmatrix}$. Then, it is easy to check that $\check{Z}_{\bar{o}}(t) = \mathbf{0}$ and $\check{Z}_{o\bar{o}}(t) = \mathbf{0}$, $\forall t \in [t_0, t_f]$, and $\check{Z}_o$ satisfies the generalized Riccati differential equation

$$\dot{\check{Z}}_o + \bar{A}'_o\check{Z}_o + \check{Z}_o\bar{A}_o - \check{Z}_o\bar{S}_o\check{Z}_o + \bar{Q}_o = \mathbf{0};\quad \check{Z}_o(t_f) = \mathbf{0} \quad (34)$$

Then, $\check{\bar{Z}}_\infty = \lim_{t_f\uparrow\infty} \check{Z}(t;t_f) = \begin{bmatrix} \lim_{t_f\uparrow\infty} \check{Z}_o(t;t_f) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} =: \begin{bmatrix} \check{\bar{Z}}_o & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ and $\check{\bar{Z}}_o$ is positive semi-definite. By the stability of the matrix $\hat{\bar{A}}_{f1}$, we have the matrix $\begin{bmatrix} \bar{A}_o - B_oR^{-1}B'_o\check{\bar{Z}}_o & \mathbf{0} \\ \bar{A}_{\bar{o}o} - B_{\bar{o}}R^{-1}B'_o\check{\bar{Z}}_o & \bar{A}_{\bar{o}} \end{bmatrix}$ is Hurwitz, and therefore the matrix $\bar{A}_o - B_oR^{-1}B'_o\check{\bar{Z}}_o$ is Hurwitz. By the observability of the pair $(\bar{A}_o, \bar{Q}_o)$, we have that the pair $(\bar{A}_o - B_oR^{-1}B'_o\check{\bar{Z}}_o, \bar{Q}_o + \check{\bar{Z}}_oB_oR^{-1}B'_o\check{\bar{Z}}_o)$ is observable, and then the pair $(\bar{A}_o - B_oR^{-1}B'_o\check{\bar{Z}}_o, \bar{Q}_o + \check{\bar{Z}}_o(B_oR^{-1}B'_o + \gamma^{-2}D_oD'_o)\check{\bar{Z}}_o)$ is observable. Note that $\check{\bar{Z}}_o$ satisfies the generalized algebraic Riccati equation

$$\bar{A}'_o\check{\bar{Z}}_o + \check{\bar{Z}}_o\bar{A}_o - \check{\bar{Z}}_o\bar{S}_o\check{\bar{Z}}_o + \bar{Q}_o = \mathbf{0} \quad (35)$$

which can be written as

$$(\bar{A}_o - B_o R^{-1} B_o' \check{\bar{Z}}_o)' \check{\bar{Z}}_o + \check{\bar{Z}}_o (\bar{A}_o - B_o R^{-1} B_o' \check{\bar{Z}}_o)$$
$$+ \check{\bar{Z}}_o (B_o R^{-1} B_o' + \gamma^{-2} D_o D_o') \check{\bar{Z}}_o + \bar{Q}_o = \mathbf{0}$$

and hence $\check{\bar{Z}}_o$ is a positive definite matrix.

The inverse of $\check{\bar{Z}}_o$ then satisfy the following Riccati equation:

$$-\check{\bar{Z}}_o^{-1} \bar{A}_o' - \bar{A}_o \check{\bar{Z}}_o^{-1} - \check{\bar{Z}}_o^{-1} \bar{Q}_o \check{\bar{Z}}_o^{-1} + \bar{S}_o = \mathbf{0} \tag{36}$$

Since $\check{\bar{Z}}_o = \lim_{t_f \uparrow \infty} \bar{Z}_o(t; t_f)$ is the least positive semi-definite solution to (35), then $\check{\bar{Z}}_o^{-1}$ is the maximal solution to the Riccati equation (36). (Since any solution $\tilde{\bar{Z}}_o$ to Riccati equation (36) with $\tilde{\bar{Z}}_o \geq \check{\bar{Z}}_o^{-1}$, then it is positive definite and invertible such that $\tilde{\bar{Z}}_o^{-1}$ satisfies generalized algebraic Riccati equation (35). Now, $\mathbf{0} < \tilde{\bar{Z}}_o^{-1} \leq \check{\bar{Z}}_o$, by the minimality of $\check{\bar{Z}}_o$, we have $\tilde{\bar{Z}}_o^{-1} = \check{\bar{Z}}_o$. Hence the maximality.) Since the pair $(\bar{A}_o, \bar{Q}_o)$ is observable, then the pair $(-\bar{A}_o', -\bar{Q}_o)$ is controllable. By (i) of Theorem 13.11 of Zhou et al. (1996), we have the matrix $-\bar{A}_o' - \bar{Q}_o \check{\bar{Z}}_o^{-1}$ has all eigenvalues in the closed left-half of the complex plane. Then, by the fact that $\check{\bar{Z}}_o^{-1}(-\bar{A}_o' - \bar{Q}_o \check{\bar{Z}}_o^{-1}) \check{\bar{Z}}_o = \bar{A}_o - \bar{S}_o \check{\bar{Z}}_o$, we have the matrix $\bar{A}_o - \bar{S}_o \check{\bar{Z}}_o$ has all eigenvalues inside $\overline{\mathbb{C}_-}$. This then implies that the matrix $\begin{bmatrix} \bar{A}_o - \bar{S}_o \check{\bar{Z}}_o & \mathbf{0} \\ \bar{A}_{\bar{o}o} - \bar{S}_{o\bar{o}}' \check{\bar{Z}}_o & \bar{A}_{\bar{o}} \end{bmatrix}$ has all eigenvalues inside $\overline{\mathbb{C}_-}$. Thus, the matrix $\bar{A} - \bar{S}\bar{Z}_\infty$ has all eigenvalue inside $\overline{\mathbb{C}_-}$. By Z_∞ be the stabilizing solution to the generalized algebraic Riccati equation (31) and Proposition 2, the Hamiltonian H_c has no eigenvalue on the imaginary axis and exactly n eigenvalues in $\mathbb{C}_-$ and the other n eigenvalues in $\mathbb{C}_+$. Then, $\bar{Z}_\infty$ being a solution to the generalized algebraic Riccati equation (31), by Theorem 13.2 of Zhou et al. (1996), the matrix $\bar{A} - \bar{S}\bar{Z}_\infty$ must be either Hurwitz or has at least one eigenvalue inside $\mathbb{C}_+$. Thus, we conclude that the matrix $\bar{A} - \bar{S}\bar{Z}_\infty$ must be Hurwitz, and $\bar{Z}_\infty = Z_\infty$.

(iv) Let the pair $(\bar{A}, \bar{Q})$ has no unobservable mode in $\overline{\mathbb{C}_-}$ and $Z_\infty \in \mathcal{S}_{\text{psd}\, n}$. Then, by (ii), $\bar{A}_{f1}$ is Hurwitz. It is easy to prove that if the pair $(\bar{A}, \bar{Q} + \gamma^{-2} Z_\infty D D' Z_\infty)$ has an unobservable mode $\lambda \in \overline{\mathbb{C}_-}$, then λ is also an unobservable mode for the pair $(\bar{A}, \bar{Q})$, which is a contradiction. Therefore, the pair $(\bar{A}, \bar{Q} + \gamma^{-2} Z_\infty D D' Z_\infty)$ has no unobservable mode in $\overline{\mathbb{C}_-}$. By Theorem 13.7 of Zhou et al. (1996), we have the algebraic Riccati equation (32) admits the stabilizing solution $\hat{Z} = Z_\infty \in \mathcal{S}_{+\, n}$.

(v) By (ii), the pair $(\bar{A}, \bar{\Pi})$ has no unobservable mode on the imaginary axis, where $\bar{\Pi} := \bar{Q} + \gamma^{-2} Z_\infty D D' Z_\infty$. Then, we will partition the pair $(\bar{A}, \bar{\Pi})$ into observable and unobservable parts, and further partition the unobservable part into stable and anti-stable parts, and lump the anti-stable unobservable part into the observable part. There exists a $T \in \mathbb{R}^{n \times n}$ and T is invertible such that $T^{-1} \bar{A} T = \begin{bmatrix} A_{o1} & \mathbf{0} \\ A_{\bar{o}oS} & A_{\bar{o}S} \end{bmatrix}$, $T' \bar{\Pi} T = \begin{bmatrix} \Pi_{o1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, where the matrix $A_{\bar{o}S}$ contains all the unobservable but stable modes of the pair $(\bar{A}, \bar{\Pi})$, and A_{o1} contains all the observable modes and unobservable and unstable modes of the pair $(\bar{A}, \bar{\Pi})$. The matrix $A_{\bar{o}S}$ is Hurwitz. The pair (A_{o1}, Π_{o1}) has no unobservable mode in $\overline{\mathbb{C}_-}$. By $Z_\infty \in \mathcal{S}_{\mathrm{psd}\, n}$ and $\bar{A}_{f1}$ being Hurwitz, we have that the Hamiltonian matrix $\tilde{H}_c := \begin{bmatrix} \bar{A} & -BR^{-1}B' \\ -\bar{\Pi} & -\bar{A}' \end{bmatrix}$ is in the domain of the Riccati operator. Let $T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, where the partition is compatible with that for the pair $(\bar{A}, \bar{\Pi})$. Then, the Hamiltonian $\tilde{H}_c$ expressed in the new coordinate is $\hat{H}_c := \bar{T}^{-1}\tilde{H}_c\bar{T} = \begin{bmatrix} T^{-1}\bar{A}T & -T^{-1}BR^{-1}B'T^{-\prime} \\ -T'\bar{\Pi}T & -T'\bar{A}'T^{-\prime} \end{bmatrix} = \begin{bmatrix} A_{o1} & \mathbf{0} & -B_1R^{-1}B_1' & -B_1R^{-1}B_2' \\ A_{\bar{o}oS} & A_{\bar{o}S} & -B_2R^{-1}B_1' & -B_2R^{-1}B_2' \\ -\Pi_{o1} & \mathbf{0} & -A_{o1}' & -A_{\bar{o}oS}' \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -A_{\bar{o}S}' \end{bmatrix}$, where $\bar{T} := \begin{bmatrix} T & \mathbf{0} \\ \mathbf{0} & T^{-\prime} \end{bmatrix}$. Since $\tilde{H}_c$ is in the domain of the Riccati operator, it must have no eigenvalue on the imaginary axis and further satisfy the complementarity condition. Since $A_{\bar{o}S}$ is Hurwitz, then the stable invariant subspace of $\hat{H}_c$ must be of the form $\begin{bmatrix} X_{11} & \mathbf{0} \\ X_{12} & X_{\bar{o}S} \\ X_{21} & \mathbf{0} \\ X_{22} & \mathbf{0} \end{bmatrix}$ where $X_{\bar{o}S}$ consists of all eigenvectors and generalized eigenvectors of $A_{\bar{o}S}$. Then, $T'Z_\infty T = \begin{bmatrix} X_{21} & \mathbf{0} \\ X_{22} & \mathbf{0} \end{bmatrix} \begin{bmatrix} X_{11} & \mathbf{0} \\ X_{12} & X_{\bar{o}S} \end{bmatrix}^{-1} = \begin{bmatrix} Z_{o1} & \mathbf{0} \\ Z_{\bar{o}oS} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} Z_{o1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, where the first equality follows from the Proposition 2; the second equality follows from the structure of the left-hand-side; and the last equality follows from Proposition 2. Then, we have $\begin{bmatrix} \Pi_{o1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = T'\bar{\Pi}T = T'\bar{Q}T + \gamma^{-2}T'Z_\infty TT^{-1}DD'T^{-\prime}T'Z_\infty T = T'\bar{Q}T + \gamma^{-2}\begin{bmatrix} Z_{o1}D_1D_1'Z_{o1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, where we have plugged in the structure for Z_∞ and $T^{-1}D =: \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$ and the partitioning is compatible with that of $\bar{A}$. Hence, $T'\bar{Q}T = \begin{bmatrix} Q_{o1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$. By $Z_\infty \in \mathcal{S}_{\mathrm{psd}\, n}$, we have Z_{o1} is positive-semidefinite. By the structure of $T^{-1}\bar{A}T$ and $T'Z_\infty T$, we have that $A_{o1} - (B_1R^{-1}B_1' - \gamma^{-2}D_1D_1')Z_{o1}$ and $A_{o1} - B_1R^{-1}B_1'Z_{o1}$ are Hurwitz, since $\bar{A} - \bar{S}Z_\infty$ and $\bar{A}_{f1}$ are Hurwitz. By generalized algebraic Riccati equation (31), we have Z_{o1} satisfies the following algebraic Riccati equation:

$$A_{o1}'Z_{o1} + Z_{o1}A_{o1} - Z_{o1}B_1R^{-1}B_1'Z_{o1} + \Pi_{o1} = \mathbf{0} \tag{37}$$

Then, Z_{o1} is a positive-semidefinite stabilizing solution to algebraic Riccati equation (37). By (iv) and the pair (A_{o1}, Π_{o1}) having no unobservable mode in $\overline{\mathbb{C}_-}$, we conclude that Z_{o1} is positive definite. Substituting the expression for Π_{o1} into algebraic Riccati equation (37), we have

$$A_{o1}'Z_{o1} + Z_{o1}A_{o1} - Z_{o1}(B_1R^{-1}B_1' - \gamma^{-2}D_1D_1')Z_{o1} + Q_{o1} = \mathbf{0} \tag{38}$$

Then, Z_{o1} is a positive definite stabilizing solution to generalized algebraic Riccati equation (38). This further implies that Z_{o1}^{-1} is positive definite and $-A_{o1}' - Q_{o1}Z_{o1}^{-1}$

is Hurwitz. Thus, the pair $(-A'_{o1}, Q_{o1})$ is stabilizable, and the pair (A_{o1}, Q_{o1}) has no unobservable mode on the imaginary axis. Hence, the pair $(\begin{bmatrix} A_{o1} & \mathbf{0} \\ A_{\bar{o}oS} & A_{\bar{o}S} \end{bmatrix}, \begin{bmatrix} Q_{o1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix})$ has no unobservable mode on the imaginary axis, since $A_{\bar{o}S}$ is Hurwitz. This is equivalent to the pair $(\bar{A}, \bar{Q})$ has no unobservable mode on the imaginary axis.

This completes the proof of the proposition. □

We make the following assumption.

Assumption 4. *The generalized algebraic Riccati equations (30) and (31) admit stabilizing solutions $\bar{\Sigma}_\infty$ and Z_∞, respectively, $\bar{\Sigma}_\infty, Z_\infty \in \mathcal{S}_{\mathrm{psd}\, n}$, and the spectral radius condition is satisfied, that is all eigenvalues of $\bar{\Sigma}_\infty Z_\infty$ (which are nonnegative real to begin with) are less than γ^2.*

Then, we have the following result, which solves the infinite-horizon H^∞ optimal control problem under imperfect state measurements completely together with Theorem 9.8 of Başar and Bernhard (1995).

Theorem 4. *Consider the infinite-horizon case of the H^∞-optimal control problem under imperfect state measurements under Assumptions 1, 2, and 4, then the matrices $\bar{A}_{f1}$ and $\tilde{A}_{f1}$ are Hurwitz, the pair (A, B) is stabilizable, the pair (A, C) is detectable, the pair $(\bar{A}, \bar{Q})$ has no unobservable modes on the imaginary axis, the pair $(\tilde{A}, \tilde{M})$ has no uncontrollable modes on the imaginary axis, $\Xi_\infty := Z_\infty(I_n - \gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \in \mathcal{S}_{\mathrm{psd}\, n}$ and is the stabilizing solution to the generalized algebraic Riccati equation:*

$$\begin{aligned}
&\Xi_\infty(A + \gamma^{-2}\bar{\Sigma}_\infty Q - (B + \gamma^{-2}\bar{\Sigma}_\infty P)R^{-1}P') + (A + \gamma^{-2}\bar{\Sigma}_\infty Q - (B \\
&+\gamma^{-2}\bar{\Sigma}_\infty P)R^{-1}P')'\Xi_\infty - \Xi_\infty((B + \gamma^{-2}\bar{\Sigma}_\infty P)R^{-1}(B + \gamma^{-2}\bar{\Sigma}_\infty P)' \\
&\qquad -\gamma^{-2}(\bar{\Sigma}_\infty C' + L)N^{-1}(\bar{\Sigma}_\infty C' + L)')\Xi_\infty + Q - PR^{-1}P' = \mathbf{0}_{n\times n} \qquad (39)
\end{aligned}$$

and the upper value of the game with cost function $J_{\gamma\infty}(\mu, \nu)$ is equal to 0. A controller guaranteeing this upper value is given by

$$\begin{aligned}
\dot{\check{x}}(t) &= (A + \gamma^{-2}\bar{\Sigma}_\infty Q)\check{x}(t) + (B + \gamma^{-2}\bar{\Sigma}_\infty P)u(t) + (\bar{\Sigma}_\infty C' \\
&\quad +L)N^{-1}(y(t) - C\check{x}(t)); \quad \check{x}(-\infty) = \mathbf{0}_n \qquad (40a)\\
u(t) &= \mu_{\mathrm{opt}}(t, \mathbf{0}_n, y_{(-\infty,t]}) = -R^{-1}(B'Z_\infty + P')(I_n - \gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1}\check{x}(t) \qquad (40b)
\end{aligned}$$

Furthermore, the closed-loop system with (1) and (40) is asymptotically stable.

Proof. By Assumption 4 and (ii) and (v) of Proposition 3 and duality, the matrices $\bar{A}_{f1}$, $\bar{A}_f$, $\tilde{A}_{f1}$, and $\tilde{A}_f$ are Hurwitz, the pair (A, B) is stabilizable, the pair (A, C) is detectable, the pair $(\bar{A}, \bar{Q})$ has no unobservable modes on the imaginary axis, and the pair $(\tilde{A}, \tilde{M})$ has no uncontrollable modes on the imaginary axis. Then, $\Xi_\infty \in \mathcal{S}_{\mathrm{psd}\, n}$ is well defined and satisfies generalized algebraic Riccati equation (39) by (22).

Next, we will show that Ξ_∞ is the stabilizing solution to the generalized algebraic Riccati equation (39).

$$\begin{aligned}
&A + \gamma^{-2}\bar{\Sigma}_\infty Q - (B + \gamma^{-2}\bar{\Sigma}_\infty P)R^{-1}P' - ((B + \gamma^{-2}\bar{\Sigma}_\infty P)R^{-1}(B + \gamma^{-2}\bar{\Sigma}_\infty P)' \\
&\quad -\gamma^{-2}(\bar{\Sigma}_\infty C' + L)N^{-1}(\bar{\Sigma}_\infty C' + L)')\Xi_\infty
\end{aligned}$$

$$
\begin{aligned}
&= A+\gamma^{-2}\bar{\Sigma}_\infty Q - BR^{-1}P' - \gamma^{-2}\bar{\Sigma}_\infty PR^{-1}P' - (BR^{-1}B' + \gamma^{-2}BR^{-1}P'\bar{\Sigma}_\infty \\
&\quad +\gamma^{-2}\bar{\Sigma}_\infty PR^{-1}B' + \gamma^{-4}\bar{\Sigma}_\infty PR^{-1}P'\bar{\Sigma}_\infty - \gamma^{-2}\bar{\Sigma}_\infty C'N^{-1}C\bar{\Sigma}_\infty \\
&\quad -\gamma^{-2}\bar{\Sigma}_\infty C'N^{-1}L' - \gamma^{-2}LN^{-1}C\bar{\Sigma}_\infty - \gamma^{-2}LN^{-1}L')Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&= A+\gamma^{-2}\bar{\Sigma}_\infty Q - BR^{-1}P' - \gamma^{-2}\bar{\Sigma}_\infty PR^{-1}P' - BR^{-1}B'Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&\quad -\gamma^{-2}BR^{-1}P'\bar{\Sigma}_\infty Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&\quad -\gamma^{-2}\bar{\Sigma}_\infty PR^{-1}B'Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&\quad -\gamma^{-4}\bar{\Sigma}_\infty PR^{-1}P'\bar{\Sigma}_\infty Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&\quad +\gamma^{-2}\bar{\Sigma}_\infty C'N^{-1}C\bar{\Sigma}_\infty Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&\quad +\gamma^{-2}\bar{\Sigma}_\infty C'N^{-1}L'Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&\quad +\gamma^{-2}LN^{-1}C\bar{\Sigma}_\infty Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} + \gamma^{-2}LN^{-1}L'Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&= A+\gamma^{-2}\bar{\Sigma}_\infty Q - BR^{-1}P'(I_n+\gamma^{-2}\bar{\Sigma}_\infty Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1}) \\
&\quad -\gamma^{-2}\bar{\Sigma}_\infty PR^{-1}P'(I_n+\gamma^{-2}\bar{\Sigma}_\infty Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1}) \\
&\quad -BR^{-1}B'Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&\quad -\gamma^{-2}\bar{\Sigma}_\infty PR^{-1}B'Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&\quad +\gamma^{-2}\bar{\Sigma}_\infty C'N^{-1}C\bar{\Sigma}_\infty Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&\quad +\gamma^{-2}\bar{\Sigma}_\infty C'N^{-1}L'Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&\quad +\gamma^{-2}LN^{-1}C\bar{\Sigma}_\infty Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} + \gamma^{-2}LN^{-1}L'Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&= A - LN^{-1}C + \gamma^{-2}\bar{\Sigma}_\infty Q - BR^{-1}P'(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} - \bar{\Sigma}_\infty C'N^{-1}C \\
&\quad -\gamma^{-2}\bar{\Sigma}_\infty PR^{-1}P'(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} - BR^{-1}B'Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&\quad -\gamma^{-2}\bar{\Sigma}_\infty PR^{-1}B'Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&\quad +\bar{\Sigma}_\infty C'N^{-1}C(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&\quad +\gamma^{-2}\bar{\Sigma}_\infty C'N^{-1}L'Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&\quad +LN^{-1}C(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} + \gamma^{-2}LN^{-1}L'Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&= \tilde{A} - \bar{\Sigma}_\infty\tilde{R} - BR^{-1}P'(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&\quad -\gamma^{-2}\bar{\Sigma}_\infty PR^{-1}P'(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} - BR^{-1}B'Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&\quad -\gamma^{-2}\bar{\Sigma}_\infty PR^{-1}B'Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&\quad +\bar{\Sigma}_\infty C'N^{-1}C(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&\quad +\gamma^{-2}\bar{\Sigma}_\infty C'N^{-1}L'Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&\quad +LN^{-1}C(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} + \gamma^{-2}LN^{-1}L'Z_\infty(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&= (\tilde{A}(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty) - \bar{\Sigma}_\infty\tilde{R}(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty) - BR^{-1}P' - \gamma^{-2}\bar{\Sigma}_\infty PR^{-1}P' \\
&\quad -BR^{-1}B'Z_\infty - \gamma^{-2}\bar{\Sigma}_\infty PR^{-1}B'Z_\infty + \bar{\Sigma}_\infty C'N^{-1}C + \gamma^{-2}\bar{\Sigma}_\infty C'N^{-1}L'Z_\infty \\
&\quad +LN^{-1}C + \gamma^{-2}LN^{-1}L'Z_\infty)(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&= (\tilde{A} - \bar{\Sigma}_\infty\tilde{R} - \gamma^{-2}(\tilde{A}\bar{\Sigma}_\infty - \bar{\Sigma}_\infty\tilde{R}\bar{\Sigma}_\infty)Z_\infty - BR^{-1}P' - \gamma^{-2}\bar{\Sigma}_\infty PR^{-1}P' \\
&\quad -BR^{-1}B'Z_\infty - \gamma^{-2}\bar{\Sigma}_\infty PR^{-1}B'Z_\infty + \bar{\Sigma}_\infty C'N^{-1}C + \gamma^{-2}\bar{\Sigma}_\infty C'N^{-1}L'Z_\infty \\
&\quad +LN^{-1}C + \gamma^{-2}LN^{-1}L'Z_\infty)(I_n-\gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
&= (A + \gamma^{-2}\bar{\Sigma}_\infty Q - \gamma^{-2}(-\bar{\Sigma}_\infty\tilde{A}' - \tilde{M})Z_\infty - BR^{-1}P' - \gamma^{-2}\bar{\Sigma}_\infty PR^{-1}P'
\end{aligned}
$$

$$
\begin{aligned}
&-BR^{-1}B'Z_\infty - \gamma^{-2}\bar{\Sigma}_\infty PR^{-1}B'Z_\infty + \gamma^{-2}\bar{\Sigma}_\infty C'N^{-1}L'Z_\infty \\
&+\gamma^{-2}LN^{-1}L'Z_\infty)(I_n - \gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
=\; &(A + \gamma^{-2}\bar{\Sigma}_\infty \bar{Q} + \gamma^{-2}(\bar{\Sigma}_\infty(A' - C'N^{-1}L') + DD' - LN^{-1}L')Z_\infty - BR^{-1}P' \\
&-BR^{-1}B'Z_\infty - \gamma^{-2}\bar{\Sigma}_\infty PR^{-1}B'Z_\infty + \gamma^{-2}\bar{\Sigma}_\infty C'N^{-1}L'Z_\infty \\
&+\gamma^{-2}LN^{-1}L'Z_\infty)(I_n - \gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
=\; &(\bar{A} + \gamma^{-2}\bar{\Sigma}_\infty \bar{Q} + \gamma^{-2}\bar{\Sigma}_\infty(A' - C'N^{-1}L')Z_\infty + \gamma^{-2}DD'Z_\infty \\
&-BR^{-1}B'Z_\infty - \gamma^{-2}\bar{\Sigma}_\infty PR^{-1}B'Z_\infty + \gamma^{-2}\bar{\Sigma}_\infty C'N^{-1}L'Z_\infty)(I_n - \gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
=\; &(\bar{A} - \bar{S}Z_\infty + \gamma^{-2}\bar{\Sigma}_\infty(\bar{Q} + \bar{A}'Z_\infty))(I_n - \gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
=\; &(\bar{A} - \bar{S}Z_\infty + \gamma^{-2}\bar{\Sigma}_\infty(-Z_\infty \bar{A} + Z_\infty \bar{S} Z_\infty))(I_n - \gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
=\; &(\bar{A} - \bar{S}Z_\infty - \gamma^{-2}\bar{\Sigma}_\infty Z_\infty(\bar{A} - \bar{S}Z_\infty))(I_n - \gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \\
=\; &(I_n - \gamma^{-2}\bar{\Sigma}_\infty Z_\infty)(\bar{A} - \bar{S}Z_\infty)(I_n - \gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1}
\end{aligned}
$$

Since $\bar{A}_f = \bar{A} - \bar{S}Z_\infty$ is Hurwitz, then Ξ_∞ is the stabilizing solution to the generalized algebraic Riccati equation (39). Denote $A_f := A + \gamma^{-2}\bar{\Sigma}_\infty Q - (B + \gamma^{-2}\bar{\Sigma}_\infty P)R^{-1}P' - ((B + \gamma^{-2}\bar{\Sigma}_\infty P)R^{-1}(B + \gamma^{-2}\bar{\Sigma}_\infty P)' - \gamma^{-2}(\bar{\Sigma}_\infty C' + L)N^{-1}(\bar{\Sigma}_\infty C' + L)')\Xi_\infty = (I_n - \gamma^{-2}\bar{\Sigma}_\infty Z_\infty)\bar{A}_f(I_n - \gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1}$.

Since the pair $(\tilde{A}, \tilde{M})$ has no uncontrollable mode on the imaginary axis. Then, we can first partition the system into controllable and uncontrollable parts, and then partition the uncontrollable part into stable and anti-stable parts. Then, $\tilde{A} = \begin{bmatrix} \tilde{A}_c & \tilde{A}_{c\bar{c}} \\ \mathbf{0} & \tilde{A}_{\bar{c}} \end{bmatrix}$ and $\tilde{M} = \begin{bmatrix} \tilde{M}_c & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, where the pair $(\tilde{A}_c, \tilde{M}_c)$ has no uncontrollable mode in $\overline{\mathbb{C}_-}$ and the matrix $\tilde{A}_{\bar{c}}$ is Hurwitz. Partition $\tilde{R} = \begin{bmatrix} \tilde{R}_c & \tilde{R}_{c\bar{c}} \\ \tilde{R}'_{c\bar{c}} & \tilde{R}_{\bar{c}} \end{bmatrix}$, $A = \begin{bmatrix} A_c & A_{c\bar{c}} \\ A_{\bar{c}c} & A_{\bar{c}} \end{bmatrix}$, $B = \begin{bmatrix} B_c \\ B_{\bar{c}} \end{bmatrix}$, $D = \begin{bmatrix} D_c \\ D_{\bar{c}} \end{bmatrix}$, $C = \begin{bmatrix} C_c & C_{\bar{c}} \end{bmatrix}$, $Q = \begin{bmatrix} Q_c & Q_{c\bar{c}} \\ Q'_{c\bar{c}} & Q_{\bar{c}} \end{bmatrix}$, $\bar{Q} = \begin{bmatrix} \bar{Q}_c & \bar{Q}_{c\bar{c}} \\ \bar{Q}'_{c\bar{c}} & \bar{Q}_{\bar{c}} \end{bmatrix}$, and $P = \begin{bmatrix} P_c \\ P_{\bar{c}} \end{bmatrix}$, accordingly. Then, the solution to generalized algebraic Riccati equation (30) satisfies $\bar{\Sigma}_\infty = \begin{bmatrix} \bar{\Sigma}_{c\infty} & \bar{\Sigma}_{c\bar{c}\infty} \\ \bar{\Sigma}'_{c\bar{c}\infty} & \bar{\Sigma}_{\bar{c}\infty} \end{bmatrix}$, where the partitioning is compatible with that of $\tilde{A}$. The Hamiltonian for the generalized algebraic Riccati equation (30) is of the following structure:

$$
H_o = \begin{bmatrix} \tilde{A}' & -\tilde{R} \\ -\tilde{M} & -\tilde{A} \end{bmatrix} = \begin{bmatrix} \tilde{A}'_c & \mathbf{0} & -\tilde{R}_c & -\tilde{R}_{c\bar{c}} \\ \tilde{A}'_{c\bar{c}} & \tilde{A}'_{\bar{c}} & -\tilde{R}'_{c\bar{c}} & -\tilde{R}_{\bar{c}} \\ -\tilde{M}_c & \mathbf{0} & -\tilde{A}_c & -\tilde{A}_{c\bar{c}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\tilde{A}_{\bar{c}} \end{bmatrix}
$$

Since $\tilde{A}_{\bar{c}}$ is Hurwitz, then the stable invariant subspace of H_o is spanned by the column vectors of $\begin{bmatrix} X_{1c} & \mathbf{0} \\ X_{1\bar{c}c} & X_{1\bar{c}} \\ X_{2c} & \mathbf{0} \\ X_{2\bar{c}c} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{2n\times n}$, where the dimension of $X_{1\bar{c}}$ is equal to the dimension of $\tilde{A}_{\bar{c}}$, and the matrix $X_1 := \begin{bmatrix} X_{1c} & \mathbf{0} \\ X_{1\bar{c}c} & X_{1\bar{c}} \end{bmatrix}$ is invertible by Proposition 2 and the fact that generalized algebraic Riccati equation (30) admits the stabilizing solution $\bar{\Sigma}_\infty$. The stabilizing solution to generalized algebraic Riccati equation (30) is given by $\bar{\Sigma}_\infty = X_2 X_1^{-1} = \begin{bmatrix} X_{2c} & \mathbf{0} \\ X_{2\bar{c}c} & \mathbf{0} \end{bmatrix} \begin{bmatrix} X_{1c} & \mathbf{0} \\ X_{1\bar{c}c} & X_{1\bar{c}} \end{bmatrix}^{-1} = \begin{bmatrix} \bar{\Sigma}_{c\infty} & \mathbf{0} \\ \bar{\Sigma}'_{c\bar{c}\infty} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \bar{\Sigma}_{c\infty} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$. Then $\bar{\Sigma}_{c\infty}$ is the stabilizing solution to the generalized algebraic Riccati equation:

$$\tilde{A}_c \bar{\Sigma}_{c\infty} + \bar{\Sigma}_{c\infty} \tilde{A}'_c - \bar{\Sigma}_{c\infty} \tilde{R}_c \bar{\Sigma}_{c\infty} + \tilde{M}_c = \mathbf{0} \tag{41}$$

By $\bar{\Sigma}_\infty \in \mathcal{S}_{\mathrm{psd}\, n}$, we have $\bar{\Sigma}_{c\infty}$ is positive semi-definite. By (iv) of Proposition 3 and duality, we have $\bar{\Sigma}_{c\infty}$ is a positive definite matrix. By (ii) of Proposition 3 and duality, we have the pair (A_c, C_c) is detectable.

Next, we will show that the system (1) and the optimal control strategy (40) form an asymptotically stable system. This then implies that μ_{opt} is an admissible controller. We express the closed-loop system with (1) and (40) in the $\begin{bmatrix} \tilde{x}' & \check{x}' \end{bmatrix}'$ coordinates as, where $\tilde{x} := x - \check{x}$,

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{x}} \\ \dot{\check{x}} \end{bmatrix} &= \begin{bmatrix} \tilde{A}_{f1} & -\gamma^{-2}\bar{\Sigma}_\infty Q \\ (\bar{\Sigma}_\infty C' + L) N^{-1} C & A + \gamma^{-2}\bar{\Sigma}_\infty Q \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \check{x} \end{bmatrix} + \begin{bmatrix} -\gamma^{-2}\bar{\Sigma}_\infty P \\ B + \gamma^{-2}\bar{\Sigma}_\infty P \end{bmatrix} u \\ &\quad + \begin{bmatrix} D - (\bar{\Sigma}_\infty C' + L) N^{-1} E \\ (\bar{\Sigma}_\infty C' + L) N^{-1} E \end{bmatrix} w =: \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \check{x} \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u + \begin{bmatrix} \bar{D}_1 \\ \bar{D}_2 \end{bmatrix} w \end{aligned} \tag{42}$$

Based on the partitioning of the system, we have

$$\begin{aligned} F_{11} &= \begin{bmatrix} \tilde{A}_c - \bar{\Sigma}_{c\infty} C'_c N^{-1} C_c & \tilde{A}_{c\bar{c}} - \bar{\Sigma}_{c\infty} C'_c N^{-1} C_{\bar{c}} \\ \mathbf{0} & \tilde{A}_{\bar{c}} \end{bmatrix}; \quad \bar{D}_2 = \begin{bmatrix} (\bar{\Sigma}_{c\infty} C'_c + L_c) N^{-1} E \\ L_{\bar{c}} N^{-1} E \end{bmatrix} \\ F_{12} &= \begin{bmatrix} -\gamma^{-2}\bar{\Sigma}_{c\infty} Q_c & -\gamma^{-2}\bar{\Sigma}_{c\infty} Q_{c\bar{c}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}; \quad \bar{B}_2 = \begin{bmatrix} B_c + \gamma^{-2}\bar{\Sigma}_{c\infty} P_c \\ B_{\bar{c}} \end{bmatrix} \\ F_{21} &= \begin{bmatrix} (\bar{\Sigma}_{c\infty} C'_c + L_c) N^{-1} C_c & (\bar{\Sigma}_{c\infty} C'_c + L_c) N^{-1} C_{\bar{c}} \\ L_{\bar{c}} N^{-1} C_c & L_{\bar{c}} N^{-1} C_{\bar{c}} \end{bmatrix}; \quad \bar{B}_1 = \begin{bmatrix} -\gamma^{-2}\bar{\Sigma}_{c\infty} P_c \\ \mathbf{0} \end{bmatrix} \\ F_{22} &= \begin{bmatrix} A_c + \gamma^{-2}\bar{\Sigma}_{c\infty} Q_c & A_{c\bar{c}} + \gamma^{-2}\bar{\Sigma}_{c\infty} Q_{c\bar{c}} \\ A_{\bar{c}c} & A_{\bar{c}} \end{bmatrix}; \quad \bar{D}_1 = \begin{bmatrix} D_c - (\bar{\Sigma}_{c\infty} C'_c + L_c) N^{-1} E \\ D_{\bar{c}} - L_{\bar{c}} N^{-1} E \end{bmatrix} \end{aligned}$$

Partition compatibly $\tilde{x} = (\tilde{x}_c, \tilde{x}_{\bar{c}})$. We observe that the $\tilde{x}_{\bar{c}}$ dynamics is simply $\dot{\tilde{x}}_{\bar{c}} = \tilde{A}_{\bar{c}} \tilde{x}_{\bar{c}} + (D_{\bar{c}} - L_{\bar{c}} N^{-1} E) w$, which is asymptotically stable. Then, the asymptotic stability of the closed-loop system is equivalent to the asymptotic stability of the matrix

$$\begin{aligned} &\begin{bmatrix} \tilde{A}_c - \bar{\Sigma}_{c\infty} C'_c N^{-1} C_c & -\gamma^{-2}\bar{\Sigma}_{c\infty} Q_c & -\gamma^{-2}\bar{\Sigma}_{c\infty} Q_{c\bar{c}} \\ (\bar{\Sigma}_{c\infty} C'_c + L_c) N^{-1} C_c & A_c + \gamma^{-2}\bar{\Sigma}_{c\infty} Q_c & A_{c\bar{c}} + \gamma^{-2}\bar{\Sigma}_{c\infty} Q_{c\bar{c}} \\ L_{\bar{c}} N^{-1} C_c & A_{\bar{c}c} & A_{\bar{c}} \end{bmatrix} \\ &\quad + \begin{bmatrix} -\gamma^{-2}\bar{\Sigma}_{c\infty} P_c \\ B_c + \gamma^{-2}\bar{\Sigma}_{c\infty} P_c \\ B_{\bar{c}} \end{bmatrix} \begin{bmatrix} \mathbf{0} & -R^{-1}(B' Z_\infty + P')(I_n - \gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \end{bmatrix} =: \hat{F} + \hat{B} K_{\mathrm{opt}} \end{aligned}$$

Let

$$\hat{D} := \begin{bmatrix} D_c - (\bar{\Sigma}_{c\infty}C_c' + L_c)N^{-1}E \\ (\bar{\Sigma}_{c\infty}C_c' + L_c)N^{-1}E \\ L_{\bar{c}}N^{-1}E \end{bmatrix}$$

We will show by brute force that $\hat{\Gamma} := \begin{bmatrix} \gamma^2\bar{\Sigma}_{c\infty}^{-1} & \mathbf{0} \\ \mathbf{0} & \Xi_\infty \end{bmatrix}$, which is a positive semi-definite matrix, is the solution to the generalized algebraic Riccati equation

$$(\hat{F} - \hat{B}R^{-1}\begin{bmatrix} P_c \\ P_c \\ P_{\bar{c}} \end{bmatrix}')'\hat{\Gamma} + \hat{\Gamma}(\hat{F} - \hat{B}R^{-1}\begin{bmatrix} P_c \\ P_c \\ P_{\bar{c}} \end{bmatrix}') - \hat{\Gamma}(\hat{B}R^{-1}\hat{B}' - \gamma^{-2}\hat{D}\hat{D}')\hat{\Gamma} + \begin{bmatrix} \bar{Q}_c & \bar{Q}_c & \bar{Q}_{c\bar{c}} \\ \bar{Q}_c & \bar{Q}_c & \bar{Q}_{c\bar{c}} \\ \bar{Q}_{c\bar{c}}' & \bar{Q}_{c\bar{c}}' & \bar{Q}_{\bar{c}} \end{bmatrix} = \mathbf{0} \quad (43)$$

$$\begin{aligned}
\text{11-block} &= (\tilde{A}_c - \bar{\Sigma}_{c\infty}C_c'N^{-1}C_c + \gamma^{-2}\bar{\Sigma}_{c\infty}P_cR^{-1}P_c')'\gamma^2\bar{\Sigma}_{c\infty}^{-1} + \gamma^2\bar{\Sigma}_{c\infty}^{-1}(\tilde{A}_c \\
&\quad -\bar{\Sigma}_{c\infty}C_c'N^{-1}C_c + \gamma^{-2}\bar{\Sigma}_{c\infty}P_cR^{-1}P_c') - \gamma^2\bar{\Sigma}_{c\infty}^{-1}(\gamma^{-4}\bar{\Sigma}_{c\infty}P_cR^{-1}P_c'\bar{\Sigma}_{c\infty} \\
&\quad -\gamma^{-2}(D_c - (\bar{\Sigma}_{c\infty}C_c' + L_c)N^{-1}E)(D_c' - E'N^{-1}(C_c\bar{\Sigma}_{c\infty} + L_c')))\gamma^2\bar{\Sigma}_{c\infty}^{-1} + \bar{Q}_c \\
&= \gamma^2\tilde{A}_c'\bar{\Sigma}_{c\infty}^{-1} - \gamma^2C_c'N^{-1}C_c + P_cR^{-1}P_c' + \gamma^2\bar{\Sigma}_{c\infty}^{-1}\tilde{A}_c - \gamma^2C_c'N^{-1}C_c + P_cR^{-1}P_c' \\
&\quad -P_cR^{-1}P_c' + \gamma^2\bar{\Sigma}_{c\infty}^{-1}(D_c - (\bar{\Sigma}_{c\infty}C_c' + L_c)N^{-1}E)(D_c' \\
&\quad -E'N^{-1}(C_c\bar{\Sigma}_{c\infty} + L_c'))\bar{\Sigma}_{c\infty}^{-1} + \bar{Q}_c \\
&= \gamma^2\tilde{A}_c'\bar{\Sigma}_{c\infty}^{-1} - \gamma^2C_c'N^{-1}C_c + \gamma^2\bar{\Sigma}_{c\infty}^{-1}\tilde{A}_c - \gamma^2C_c'N^{-1}C_c + P_cR^{-1}P_c' + \bar{Q}_c \\
&\quad +\gamma^2\bar{\Sigma}_{c\infty}^{-1}D_cD_c'\bar{\Sigma}_{c\infty}^{-1} - \gamma^2\bar{\Sigma}_{c\infty}^{-1}D_cE'N^{-1}(C_c\bar{\Sigma}_{c\infty} + L_c')\bar{\Sigma}_{c\infty}^{-1} - \gamma^2\bar{\Sigma}_{c\infty}^{-1}(\bar{\Sigma}_{c\infty}C_c' \\
&\quad +L_c)N^{-1}ED_c'\bar{\Sigma}_{c\infty}^{-1} + \gamma^2\bar{\Sigma}_{c\infty}^{-1}(\bar{\Sigma}_{c\infty}C_c' + L_c)N^{-1}EE'N^{-1}(C_c\bar{\Sigma}_{c\infty} + L_c')\bar{\Sigma}_{c\infty}^{-1} \\
&= \gamma^2\tilde{A}_c'\bar{\Sigma}_{c\infty}^{-1} - \gamma^2C_c'N^{-1}C_c + \gamma^2\bar{\Sigma}_{c\infty}^{-1}\tilde{A}_c - \gamma^2C_c'N^{-1}C_c + P_cR^{-1}P_c' + \bar{Q}_c \\
&\quad +\gamma^2\bar{\Sigma}_{c\infty}^{-1}D_cD_c'\bar{\Sigma}_{c\infty}^{-1} - \gamma^2\bar{\Sigma}_{c\infty}^{-1}L_cN^{-1}C_c - \gamma^2\bar{\Sigma}_{c\infty}^{-1}L_cN^{-1}L_c'\bar{\Sigma}_{c\infty}^{-1} - \gamma^2C_c'N^{-1}L_c'\bar{\Sigma}_{c\infty}^{-1} \\
&\quad -\gamma^2\bar{\Sigma}_{c\infty}^{-1}L_cN^{-1}L_c'\bar{\Sigma}_{c\infty}^{-1} + \gamma^2\bar{\Sigma}_{c\infty}^{-1}(\bar{\Sigma}_{c\infty}C_c' + L_c)N^{-1}(C_c\bar{\Sigma}_{c\infty} + L_c')\bar{\Sigma}_{c\infty}^{-1} \\
&= \gamma^2\tilde{A}_c'\bar{\Sigma}_{c\infty}^{-1} - \gamma^2C_c'N^{-1}C_c + \gamma^2\bar{\Sigma}_{c\infty}^{-1}\tilde{A}_c - \gamma^2\tilde{R}_c \\
&\quad +\gamma^2\bar{\Sigma}_{c\infty}^{-1}D_cD_c'\bar{\Sigma}_{c\infty}^{-1} - \gamma^2\bar{\Sigma}_{c\infty}^{-1}L_cN^{-1}L_c'\bar{\Sigma}_{c\infty}^{-1} \\
&\quad -\gamma^2\bar{\Sigma}_{c\infty}^{-1}L_cN^{-1}L_c'\bar{\Sigma}_{c\infty}^{-1} + \gamma^2C_c'N^{-1}C_c + \gamma^2\bar{\Sigma}_{c\infty}^{-1}L_cN^{-1}L_c'\bar{\Sigma}_{c\infty}^{-1} \\
&= \gamma^2\tilde{A}_c'\bar{\Sigma}_{c\infty}^{-1} + \gamma^2\bar{\Sigma}_{c\infty}^{-1}\tilde{A}_c - \gamma^2\tilde{R}_c + \gamma^2\bar{\Sigma}_{c\infty}^{-1}\tilde{M}_c\bar{\Sigma}_{c\infty}^{-1} \\
&= \gamma^2\bar{\Sigma}_{c\infty}^{-1}(\bar{\Sigma}_{c\infty}\tilde{A}_c' + \tilde{A}_c\bar{\Sigma}_{c\infty} - \bar{\Sigma}_{c\infty}\tilde{R}_c\bar{\Sigma}_{c\infty} + \tilde{M}_c)\bar{\Sigma}_{c\infty}^{-1} = \mathbf{0}
\end{aligned}$$

12 & 13-block

$$\begin{aligned}
&= \gamma^2\bar{\Sigma}_{c\infty}^{-1}\left[\, -\gamma^{-2}\bar{\Sigma}_{c\infty}Q_c + \gamma^{-2}\bar{\Sigma}_{c\infty}P_cR^{-1}P_c' \;\; -\gamma^{-2}\bar{\Sigma}_{c\infty}Q_{c\bar{c}} + \gamma^{-2}\bar{\Sigma}_{c\infty}P_cR^{-1}P_{\bar{c}}' \,\right] \\
&\quad + \left[\, C_c'N^{-1}(L_c' + C_c\bar{\Sigma}_{c\infty}) - P_cR^{-1}(B_c' + \gamma^{-2}P_c'\bar{\Sigma}_{c\infty}) \;\; C_c'N^{-1}L_{\bar{c}}' - P_cR^{-1}B_{\bar{c}}' \,\right]\Xi_\infty \\
&\quad -\gamma^2\bar{\Sigma}_{c\infty}^{-1}(\left[\, -\gamma^{-2}\bar{\Sigma}_{c\infty}P_cR^{-1}(B_c' + \gamma^{-2}P_c'\bar{\Sigma}_{c\infty}) \;\; -\gamma^{-2}\bar{\Sigma}_{c\infty}P_cR^{-1}B_{\bar{c}}' \,\right] \\
&\quad -\gamma^{-2}\begin{bmatrix} (D_c - (\bar{\Sigma}_{c\infty}C_c' + L_c)N^{-1}E) & (D_c - (\bar{\Sigma}_{c\infty}C_c' + L_c)N^{-1}E) \\ \cdot E'N^{-1}(L_c' + C_c\bar{\Sigma}_{c\infty}) & \cdot E'N^{-1}L_{\bar{c}}' \end{bmatrix})\Xi_\infty
\end{aligned}$$

$$
\begin{aligned}
&+\left[\, \bar{Q}_c \;\; \bar{Q}_{c\bar{c}} \,\right] \\
=&\left[\, -Q_c + P_c R^{-1} P_c' \;\; -Q_{c\bar{c}} + P_c R^{-1} P_{\bar{c}}' \,\right] + \left[\, \bar{Q}_c \;\; \bar{Q}_{c\bar{c}} \,\right] \\
&+\left[\, C_c' N^{-1}(L_c' + C_c \bar{\Sigma}_{c\infty}) - P_c R^{-1}(B_c' + \gamma^{-2} P_c' \bar{\Sigma}_{c\infty}) \;\; C_c' N^{-1} L_{\bar{c}}' - P_c R^{-1} B_{\bar{c}}' \,\right] \Xi_\infty \\
&+\left[\, P_c R^{-1}(B_c' + \gamma^{-2} P_c' \bar{\Sigma}_{c\infty}) \;\; P_c R^{-1} B_{\bar{c}}' \,\right] \Xi_\infty \\
&+\bar{\Sigma}_{c\infty}^{-1} \begin{bmatrix} L_c N^{-1}(L_c' + C_c \bar{\Sigma}_{c\infty}) & L_c N^{-1} L_{\bar{c}}' - (\bar{\Sigma}_{c\infty} C_c' \\ -(\bar{\Sigma}_{c\infty} C_c' + L_c) N^{-1}(L_c' + C_c \bar{\Sigma}_{c\infty}) & +L_c) N^{-1} L_{\bar{c}}' \end{bmatrix} \Xi_\infty \\
=&\left[\, C_c' N^{-1}(L_c' + C_c \bar{\Sigma}_{c\infty}) \;\; C_c' N^{-1} L_{\bar{c}}' \,\right] \Xi_\infty \\
&+\left[\, -C_c' N^{-1}(L_c' + C_c \bar{\Sigma}_{c\infty}) \;\; -C_c' N^{-1} L_{\bar{c}}' \,\right] \Xi_\infty = \mathbf{0}
\end{aligned}
$$

21 & 31-block $= \mathbf{0}$ by symmetry

$$
\begin{aligned}
&\begin{bmatrix} 22 & 23 \\ 32 & 33 \end{bmatrix}\text{-block} = (A + \gamma^{-2} \bar{\Sigma}_\infty Q - \bar{B}_2 R^{-1} P')' \Xi_\infty + \Xi_\infty (A + \gamma^{-2} \bar{\Sigma}_\infty Q - \bar{B}_2 R^{-1} P') \\
&\quad -\Xi_\infty (\bar{B}_2 R^{-1} \bar{B}_2' - \gamma^{-2} \bar{D}_2 \bar{D}_2') \Xi_\infty + \bar{Q} \\
&= (A + \gamma^{-2} \bar{\Sigma}_\infty Q - (B + \gamma^{-2} \bar{\Sigma}_\infty P) R^{-1} P')' \Xi_\infty \\
&\quad +\Xi_\infty (A + \gamma^{-2} \bar{\Sigma}_\infty Q - (B + \gamma^{-2} \bar{\Sigma}_\infty P) R^{-1} P') \\
&\quad -\Xi_\infty ((B + \gamma^{-2} \bar{\Sigma}_\infty P) R^{-1} (B + \gamma^{-2} \bar{\Sigma}_\infty P)' \\
&\quad -\gamma^{-2} ((\bar{\Sigma}_\infty C' + L) N^{-1} E)((\bar{\Sigma}_\infty C' + L) N^{-1} E)') \Xi_\infty + \bar{Q} = \text{LHS of (39)} = \mathbf{0}_{n\times n}
\end{aligned}
$$

Hence, $\hat{\Gamma}$ is a positive semi-definite solution to generalized algebraic Riccati equation (43). We will show that $\hat{\Gamma}$ is not the stabilizing solution to (43). Toward this end, we compute the matrix $\hat{F} - \hat{B} R^{-1} \begin{bmatrix} P_c \\ P_c \\ P_{\bar{c}} \end{bmatrix}' - (\hat{B} R^{-1} \hat{B}' - \gamma^{-2} \hat{D} \hat{D}') \hat{\Gamma}$.

$$
\begin{aligned}
\text{11-block} &= \tilde{A}_c - \bar{\Sigma}_{c\infty} C_c' N^{-1} C_c + \gamma^{-2} \bar{\Sigma}_{c\infty} P_c R^{-1} P_c' - (\gamma^{-4} \bar{\Sigma}_{c\infty} P_c R^{-1} P_c' \bar{\Sigma}_{c\infty} \\
&\quad -\gamma^{-2} (D_c - (\bar{\Sigma}_{c\infty} C_c' + L_c) N^{-1} E)(D_c' - E' N^{-1} (C_c \bar{\Sigma}_{c\infty} + L_c'))) \gamma^2 \bar{\Sigma}_{c\infty}^{-1} \\
&= \tilde{A}_c - \bar{\Sigma}_{c\infty} C_c' N^{-1} C_c + D_c D_c' \bar{\Sigma}_{c\infty}^{-1} - (\bar{\Sigma}_{c\infty} C_c' + L_c) N^{-1} L_c' \bar{\Sigma}_{c\infty}^{-1} \\
&\quad -L_c N^{-1} (C_c \bar{\Sigma}_{c\infty} + L_c') \bar{\Sigma}_{c\infty}^{-1} + (\bar{\Sigma}_{c\infty} C_c' + L_c) N^{-1} (C_c \bar{\Sigma}_{c\infty} + L_c') \bar{\Sigma}_{c\infty}^{-1} \\
&= \tilde{A}_c - \bar{\Sigma}_{c\infty} C_c' N^{-1} C_c + D_c D_c' \bar{\Sigma}_{c\infty}^{-1} \\
&\quad -L_c N^{-1} (C_c \bar{\Sigma}_{c\infty} + L_c') \bar{\Sigma}_{c\infty}^{-1} + (\bar{\Sigma}_{c\infty} C_c' + L_c) N^{-1} C_c \\
&= \tilde{A}_c + \tilde{M}_c \bar{\Sigma}_{c\infty}^{-1} = \bar{\Sigma}_{c\infty} (-\tilde{A}_c' + \tilde{R}_c \bar{\Sigma}_{c\infty}) \bar{\Sigma}_{c\infty}^{-1} = -\bar{\Sigma}_{c\infty} (\tilde{A}_c' - \tilde{R}_c \bar{\Sigma}_{c\infty}) \bar{\Sigma}_{c\infty}^{-1}
\end{aligned}
$$

This matrix is anti-stable, since the matrix $\tilde{A}_c - \bar{\Sigma}_{c\infty} \tilde{R}_c$ is Hurwitz.

$$
\begin{aligned}
&\text{21 \& 31-block} = \begin{bmatrix} (\bar{\Sigma}_{c\infty} C_c' + L_c) N^{-1} C_c \\ L_{\bar{c}} N^{-1} C_c \end{bmatrix} - \bar{B}_2 R^{-1} P_c' - (-\gamma^{-2} \bar{B}_2 R^{-1} P_c' \bar{\Sigma}_{c\infty} \\
&\quad -\gamma^{-2} \bar{D}_2 (D_c' - E' N^{-1} (C_c \bar{\Sigma}_{c\infty} + L_c'))) \gamma^2 \bar{\Sigma}_{c\infty}^{-1} \\
&= (\bar{\Sigma}_\infty C' + L) N^{-1} C_c - \bar{B}_2 R^{-1} P_c' + \bar{B}_2 R^{-1} P_c' + ((\bar{\Sigma}_\infty C' + L) N^{-1} E)(D_c' \\
&\quad -E' N^{-1} (C_c \bar{\Sigma}_{c\infty} + L_c')) \bar{\Sigma}_{c\infty}^{-1} \\
&= (\bar{\Sigma}_\infty C' + L) N^{-1} C_c + (\bar{\Sigma}_\infty C' + L) N^{-1} L_c' \bar{\Sigma}_{c\infty}^{-1} - (\bar{\Sigma}_\infty C' + L) N^{-1} (C_c \bar{\Sigma}_{c\infty}
\end{aligned}
$$

$$+L_c')\bar{\Sigma}_{c\infty}^{-1} = \mathbf{0}$$

$$\begin{bmatrix} 22 & 23 \\ 32 & 33 \end{bmatrix}\text{-block} = A + \gamma^{-2}\bar{\Sigma}_\infty Q - \bar{B}_2 R^{-1} P' - (\bar{B}_2 R^{-1} \bar{B}_2' - \gamma^{-2}\bar{D}_2\bar{D}_2')\Xi_\infty$$

$$= A + \gamma^{-2}\bar{\Sigma}_\infty Q - (B + \gamma^{-2}\bar{\Sigma}_\infty P)R^{-1}P' - ((B + \gamma^{-2}\bar{\Sigma}_\infty P)R^{-1}(B' + \gamma^{-2}P'\bar{\Sigma}_\infty)$$
$$-\gamma^{-2}(\bar{\Sigma}_\infty C' + L)N^{-1}EE'N^{-1}(\bar{\Sigma}_\infty C' + L)')\Xi_\infty$$
$$= A + \gamma^{-2}\bar{\Sigma}_\infty Q - (B + \gamma^{-2}\bar{\Sigma}_\infty P)R^{-1}P' - (B + \gamma^{-2}\bar{\Sigma}_\infty P)R^{-1}(B' + \gamma^{-2}P'\bar{\Sigma}_\infty)\Xi_\infty$$
$$+\gamma^{-2}(\bar{\Sigma}_\infty C' + L)N^{-1}(C\bar{\Sigma}_\infty + L')\Xi_\infty = A_f$$
$$= (I_n - \gamma^{-2}\bar{\Sigma}_\infty Z_\infty)(\bar{A} - \bar{S}Z_\infty)(I_n - \gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1}$$

This shows that

$$\hat{F} - \hat{B}R^{-1}\begin{bmatrix} P_c \\ P_c \\ P_{\bar{c}} \end{bmatrix}' - (\hat{B}R^{-1}\hat{B}' - \gamma^{-2}\hat{D}\hat{D}')\hat{\Gamma} = \begin{bmatrix} -\bar{\Sigma}_{c\infty}(\tilde{A}_c' - \tilde{R}_c\bar{\Sigma}_{c\infty})\bar{\Sigma}_{c\infty}^{-1} & \star \\ \mathbf{0} & A_f \end{bmatrix}$$

where A_f is Hurwitz. Note the following

$$-R^{-1}\begin{bmatrix} P_c \\ P_c \\ P_{\bar{c}} \end{bmatrix}' - R^{-1}\hat{B}'\hat{\Gamma}$$
$$= -R^{-1}\begin{bmatrix} P_c' & P' \end{bmatrix} - R^{-1}\begin{bmatrix} -\gamma^{-2}P_c'\bar{\Sigma}_{c\infty} & B' + \gamma^{-2}P'\bar{\Sigma}_\infty \end{bmatrix}\hat{\Gamma}$$
$$= -\begin{bmatrix} R^{-1}P_c' & R^{-1}P' \end{bmatrix} - \begin{bmatrix} -\gamma^{-2}R^{-1}P_c'\bar{\Sigma}_{c\infty} & R^{-1}(B' + \gamma^{-2}P'\bar{\Sigma}_\infty) \end{bmatrix}\hat{\Gamma}$$
$$= -\begin{bmatrix} R^{-1}P_c' & R^{-1}P' \end{bmatrix} - \begin{bmatrix} -R^{-1}P_c' & R^{-1}(B' + \gamma^{-2}P'\bar{\Sigma}_\infty)\Xi_\infty \end{bmatrix}$$
$$= -\begin{bmatrix} \mathbf{0} & R^{-1}P' + R^{-1}(B' + \gamma^{-2}P'\bar{\Sigma}_\infty)\Xi_\infty \end{bmatrix}$$
$$= -\begin{bmatrix} \mathbf{0} & R^{-1}P' + R^{-1}B'\Xi_\infty + \gamma^{-2}R^{-1}P'\bar{\Sigma}_\infty Z_\infty(I_n - \gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \end{bmatrix}$$
$$= -\begin{bmatrix} \mathbf{0} & R^{-1}B'\Xi_\infty + R^{-1}P'(I_n - \gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \end{bmatrix}$$
$$= -\begin{bmatrix} \mathbf{0} & R^{-1}(B'Z_\infty + P')(I_n - \gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \end{bmatrix} = K_{\text{opt}}$$

Hence, we have $\hat{F} + \hat{B}K_{\text{opt}} = \hat{F} - \hat{B}R^{-1}\begin{bmatrix} P_c \\ P_c \\ P_{\bar{c}} \end{bmatrix}' - \hat{B}R^{-1}\hat{B}'\hat{\Gamma}$.

By $\hat{\Gamma}$ being the solution to generalized algebraic Riccati equation (43), we have

$$\hat{\Gamma}(\hat{F} + \hat{B}K_{\text{opt}}) + (\hat{F} + \hat{B}K_{\text{opt}})'\hat{\Gamma} + \hat{\Gamma}(\hat{B}R^{-1}\hat{B}' + \gamma^{-2}\hat{D}\hat{D}')\hat{\Gamma} + \begin{bmatrix} \bar{Q}_c & \bar{Q}_c & \bar{Q}_{c\bar{c}} \\ \bar{Q}_c & \bar{Q}_c & \bar{Q}_{c\bar{c}} \\ \bar{Q}_{c\bar{c}}' & \bar{Q}_{c\bar{c}}' & \bar{Q}_{\bar{c}} \end{bmatrix}$$
$$= \mathbf{0} \tag{44}$$

Now, we will show that the pair $(\hat{F} + \hat{B}K_{\text{opt}}, \hat{\Gamma}(\hat{B}R^{-1}\hat{B}' + \gamma^{-2}\hat{D}\hat{D}')\hat{\Gamma} + \begin{bmatrix} \bar{Q}_c & \bar{Q}_c & \bar{Q}_{c\bar{c}} \\ \bar{Q}_c & \bar{Q}_c & \bar{Q}_{c\bar{c}} \\ \bar{Q}'_{c\bar{c}} & \bar{Q}'_{c\bar{c}} & \bar{Q}_{\bar{c}} \end{bmatrix})$ is detectable. Let $\lambda \in \mathbb{C}$ be any unobservable mode of the pair. Then, there exists a complex vector $\begin{bmatrix} \tilde{x}'_{c1} & \check{x}'_1 \end{bmatrix}' = \begin{bmatrix} \tilde{x}'_{c1} & \check{x}'_{c1} & \check{x}_{\bar{c}1} \end{bmatrix}' \neq \mathbf{0}$ such that

$$\lambda \begin{bmatrix} \tilde{x}_{c1} \\ \check{x}_1 \end{bmatrix} = (\hat{F} + \hat{B}K_{\text{opt}}) \begin{bmatrix} \tilde{x}_{c1} \\ \check{x}_1 \end{bmatrix} \tag{45a}$$

$$\mathbf{0} = \hat{\Gamma}(\hat{B}R^{-1}\hat{B}' + \gamma^{-2}\hat{D}\hat{D}')\hat{\Gamma} + \begin{bmatrix} \bar{Q}_c & \bar{Q}_c & \bar{Q}_{c\bar{c}} \\ \bar{Q}_c & \bar{Q}_c & \bar{Q}_{c\bar{c}} \\ \bar{Q}'_{c\bar{c}} & \bar{Q}'_{c\bar{c}} & \bar{Q}_{\bar{c}} \end{bmatrix}) \begin{bmatrix} \tilde{x}_{c1} \\ \check{x}_1 \end{bmatrix} \tag{45b}$$

Now, pre-multiply $\begin{bmatrix} \overline{\tilde{x}_{c1}}' & \overline{\check{x}_1}' \end{bmatrix}'$ to (45b), and we have $\begin{bmatrix} \overline{\tilde{x}_{c1}}' & \overline{\check{x}_1}' \end{bmatrix}' \hat{\Gamma}(\hat{B}R^{-1}\hat{B}' + \gamma^{-2}\hat{D}\hat{D}')\hat{\Gamma} \begin{bmatrix} \tilde{x}_{c1} \\ \check{x}_1 \end{bmatrix} + \begin{bmatrix} \overline{\tilde{x}_{c1}}' & \overline{\check{x}_1}' \end{bmatrix}' \begin{bmatrix} \bar{Q}_c & \bar{Q}_c & \bar{Q}_{c\bar{c}} \\ \bar{Q}_c & \bar{Q}_c & \bar{Q}_{c\bar{c}} \\ \bar{Q}'_{c\bar{c}} & \bar{Q}'_{c\bar{c}} & \bar{Q}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \tilde{x}_{c1} \\ \check{x}_1 \end{bmatrix} = 0$. Note that the matrix $\begin{bmatrix} \bar{Q}_c & \bar{Q}_c & \bar{Q}_{c\bar{c}} \\ \bar{Q}_c & \bar{Q}_c & \bar{Q}_{c\bar{c}} \\ \bar{Q}'_{c\bar{c}} & \bar{Q}'_{c\bar{c}} & \bar{Q}_{\bar{c}} \end{bmatrix}$ is positive semi-definite since we have

$$\begin{bmatrix} \bar{Q}_c & \bar{Q}_c & \bar{Q}_{c\bar{c}} \\ \bar{Q}_c & \bar{Q}_c & \bar{Q}_{c\bar{c}} \\ \bar{Q}'_{c\bar{c}} & \bar{Q}'_{c\bar{c}} & \bar{Q}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} I & \mathbf{0} \\ I & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \bar{Q} \begin{bmatrix} I & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{bmatrix}$$

Then, we must have $\begin{bmatrix} \overline{\tilde{x}_{c1}}' & \overline{\check{x}_1}' \end{bmatrix}' \hat{\Gamma}\hat{B}R^{-1}\hat{B}'\hat{\Gamma} \begin{bmatrix} \tilde{x}_{c1} \\ \check{x}_1 \end{bmatrix} = \begin{bmatrix} \overline{\tilde{x}_{c1}}' & \overline{\check{x}_1}' \end{bmatrix}' \hat{\Gamma}\gamma^{-2}\hat{D}\hat{D}'\hat{\Gamma} \begin{bmatrix} \tilde{x}_{c1} \\ \check{x}_1 \end{bmatrix}$ $= \begin{bmatrix} \overline{\tilde{x}_{c1}}' & \overline{\check{x}_1}' \end{bmatrix}' \begin{bmatrix} \bar{Q}_c & \bar{Q}_c & \bar{Q}_{c\bar{c}} \\ \bar{Q}_c & \bar{Q}_c & \bar{Q}_{c\bar{c}} \\ \bar{Q}'_{c\bar{c}} & \bar{Q}'_{c\bar{c}} & \bar{Q}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \tilde{x}_{c1} \\ \check{x}_1 \end{bmatrix} = 0$ and

$$\hat{B}'\hat{\Gamma} \begin{bmatrix} \tilde{x}_{c1} \\ \check{x}_1 \end{bmatrix} = \mathbf{0}_p; \quad \gamma^{-2}\hat{D}'\hat{\Gamma} \begin{bmatrix} \tilde{x}_{c1} \\ \check{x}_1 \end{bmatrix} = \mathbf{0}_q; \quad \bar{Q} \begin{bmatrix} \tilde{x}_{c1} + \check{x}_{c1} \\ \check{x}_{\bar{c}1} \end{bmatrix} = \mathbf{0}_n \tag{46}$$

This further leads to $\lambda \begin{bmatrix} \tilde{x}_{c1} \\ \check{x}_1 \end{bmatrix} = (\hat{F} + \hat{B}K_{\text{opt}}) \begin{bmatrix} \tilde{x}_{c1} \\ \check{x}_1 \end{bmatrix} + \gamma^{-2}\hat{D}\hat{D}'\hat{\Gamma} \begin{bmatrix} \tilde{x}_{c1} \\ \check{x}_1 \end{bmatrix} = (\hat{F} - \hat{B}R^{-1} \cdot \begin{bmatrix} P_c \\ P_c \\ P_{\bar{c}} \end{bmatrix}' - (\hat{B}R^{-1}\hat{B}' - \gamma^{-2}\hat{D}\hat{D}')\hat{\Gamma}) \begin{bmatrix} \tilde{x}_{c1} \\ \check{x}_1 \end{bmatrix} = \begin{bmatrix} -\bar{\Sigma}_{c\infty}(\tilde{A}'_c - \tilde{R}_c\bar{\Sigma}_{c\infty})\bar{\Sigma}_{c\infty}^{-1} & \star \\ \mathbf{0} & A_f \end{bmatrix} \begin{bmatrix} \tilde{x}_{c1} \\ \check{x}_1 \end{bmatrix}$.

Thus, $\lambda\check{x}_1 = A_f\check{x}_1$. If $\check{x}_1 \neq \mathbf{0}_n$, we have λ is an eigenvalue of the matrix A_f with an associated eigenvector $\check{x}_1$. Since the matrix A_f is Hurwitz, then $\lambda \in \mathbb{C}_-$.

On the other hand, if $\check{x}_1 = \mathbf{0}_n$, then $\tilde{x}_{c1} \neq \mathbf{0}$ and $\lambda\tilde{x}_{c1} = -\bar{\Sigma}_{c\infty}(\tilde{A}'_c - \tilde{R}_c\bar{\Sigma}_{c\infty})\bar{\Sigma}_{c\infty}^{-1}\tilde{x}_{c1}$. Hence, λ is an eigenvalue of the matrix $-\bar{\Sigma}_{c\infty}(\tilde{A}'_c - \tilde{R}_c\bar{\Sigma}_{c\infty})\bar{\Sigma}_{c\infty}^{-1} = \tilde{A}_c + \tilde{M}_c\bar{\Sigma}_{c\infty}^{-1}$ with an associated eigenvector $\tilde{x}_{c1}$. Since the matrix $-\bar{\Sigma}_{c\infty}(\tilde{A}'_c - \tilde{R}_c\bar{\Sigma}_{c\infty})\bar{\Sigma}_{c\infty}^{-1}$ is anti-stable, then $\lambda \in \mathbb{C}_+$. By (46), we have $-P'_c\tilde{x}_{c1} = \mathbf{0}_p$, $(D'_c -$

$E'N^{-1}(L_c' + C_c\bar{\Sigma}_{c\infty}))\bar{\Sigma}_{c\infty}^{-1}\tilde{x}_{c1} = \mathbf{0}_q$, and $\bar{Q}_c\tilde{x}_{c1} = \mathbf{0}$. Left multiply E to the second equation to conclude $C_c\tilde{x}_{c1} = \mathbf{0}_m$. Now, left multiply D_c to the same equation to conclude $D_cD_c'\bar{\Sigma}_{c\infty}^{-1}\tilde{x}_{c1} = L_cN^{-1}L_c'\bar{\Sigma}_{c\infty}^{-1}\tilde{x}_{c1}$, or equivalently, $\tilde{M}_c\bar{\Sigma}_{c\infty}^{-1}\tilde{x}_{c1} = \mathbf{0}$. These further lead to $\lambda\tilde{x}_{c1} = (\tilde{A}_c + \tilde{M}_c\bar{\Sigma}_{c\infty}^{-1})\tilde{x}_{c1} = A_c\tilde{x}_{c1}$. This shows that $\lambda \in \mathbb{C}_+$ is an eigenvalue of A_c with an associated eigenvector $\tilde{x}_{c1}$ and we have $C_c\tilde{x}_{c1} = \mathbf{0}_m$. This is a contradiction to the pair (A_c, C_c) being detectable. Hence, we must have $\check{x}_1 \neq \mathbf{0}_n$.

Therefore, we conclude that $\lambda \in \mathbb{C}_-$. Then, the pair $(\hat{F} + \hat{B}K_{\text{opt}}, \hat{\Gamma}(\hat{B}R^{-1}\hat{B}' + \gamma^{-2}\hat{D}\hat{D}')\hat{\Gamma} + \begin{bmatrix} \bar{Q}_c & \bar{Q}_c & \bar{Q}_{c\bar{c}} \\ \bar{Q}_c & \bar{Q}_c & \bar{Q}_{c\bar{c}} \\ \bar{Q}_{c\bar{c}}' & \bar{Q}_{c\bar{c}}' & \bar{Q}_{\bar{c}} \end{bmatrix})$ is detectable. By $\hat{\Gamma}$ being a positive semi-definite solution to the Lyapunov equation (44), we have the matrix $\hat{F} + \hat{B}K_{\text{opt}}$ is Hurwitz. This shows that the control law (40) is internally stabilizing and therefore admissible.

By (26) and the derivation preceding (26), we have

$$
\begin{aligned}
& J_{\gamma\infty}(\mu_{\text{opt}}, \nu) = \lim_{t_0\downarrow-\infty, t_f\uparrow\infty} J_\gamma(\mu_{\text{opt}}, \nu, \check{x}(t_0), \infty I_n, \mathbf{0}_{n\times n}) \\
& \leq \lim_{t_0\downarrow-\infty, t_f\uparrow\infty} J_\gamma(\mu_{\text{opt}}, \nu, \check{x}(t_0), \bar{\Sigma}_\infty^{-1}, Z_\infty) = \lim_{t_0\downarrow-\infty, t_f\uparrow\infty} \Big(-\left| x(t_f) - (I_n - \gamma^{-2}\bar{\Sigma}_\infty \right. \\
& \quad \left. \cdot Z_\infty)^{-1}\check{x}(t_f)\right|^2_{\gamma^2\bar{\Sigma}_\infty^{-1} - Z_\infty} + \left|\check{x}(t_0)\right|^2_{\Xi_\infty} - \int_{t_0}^{t_f} \gamma^2 \left| w(t) - (I_q - E'N^{-1}E)D' \right. \\
& \quad \left. \cdot\bar{\Sigma}_\infty^{-1}(x(t) - \check{x}(t)) + E'N^{-1}(C(x(t) - \check{x}(t)) - \gamma^{-2}(C\bar{\Sigma}_\infty + L')\Xi_\infty\check{x}(t))\right|^2 \mathrm{d}t \Big) \quad (47) \\
& \leq \lim_{t_0\downarrow-\infty} \left|\check{x}(t_0)\right|^2_{\Xi_\infty} = 0
\end{aligned}
$$

where μ_{opt} is given in (40); the first equality follows from (27); the first inequality follow from the fact $\infty I_n \geq \bar{\Sigma}_\infty^{-1}$ and $\mathbf{0}_{n\times n} \leq Z_\infty$; the second equality follows from (26); and the last equality follows from $\check{x}_0 = \mathbf{0}_n$. Or, we may skip the second equality and directly arrive at the second inequality by Theorem 1.

Hence, the upper value of $J_{\gamma\infty}(\mu, \nu)$ is less than or equal to zero. The upper value is always nonnegative since the maximizer can alway play the trivial strategy ν_0: $x_0 = \mathbf{0}_n$, and $w_{(-\infty,\infty)} = \vartheta_{\bar{\mathbb{L}}_2(\mathbb{R},\mathbb{R}^q)}$ to achieve a nonnegative $J_{\gamma\infty}(\mu_{\text{opt}}, \nu_0) \geq 0$. Thus, the upper value of the game is $0 = \sup_\nu J_{\gamma\infty}(\mu_{\text{opt}}, \nu)$, and μ_{opt} achieves this upper value.

This completes the proof of the theorem. □

To obtain the solution to infinite-horizon case as the appropriate limit of the solution to the finite-horizon case, we make the following standard assumption.

Assumption 5. *The pair $(\bar{A}, \bar{Q})$ is detectable and the pair $(\tilde{A}, \tilde{M})$ is stabilizable. The generalized algebraic Riccati equations (30) and (31) admit stabilizing solutions $\bar{\Sigma}_\infty$ and Z_∞, respectively, $\bar{\Sigma}_\infty, Z_\infty \in \mathcal{S}_{\text{psd}\,n}$, and the spectral radius condition is satisfied, that is all eigenvalues of $\bar{\Sigma}_\infty Z_\infty$ (which are nonnegative real to begin with) are less than γ^2.*

Then, we have the following result.

Corollary 5. *Consider the infinite-horizon case of the H^∞-optimal control problem under imperfect state measurements under Assumptions 1, 2, and 5, then the matrices $\bar{A}_{f1}$ and $\tilde{A}_{f1}$ are Hurwitz, the pair (A, B) is stabilizable, the pair (A, C) is detectable, $\lim_{t_0 \downarrow -\infty} \bar{\Sigma}(t; t_0) \uparrow \bar{\Sigma}_\infty$ and $\lim_{t_f \uparrow \infty} Z(t; t_f) \uparrow Z_\infty$, (i. e., $\bar{\Sigma}_\infty$ and Z_∞ are the least positive semi-definite solution to generalized algebraic Riccati equations (30) and (31), respectively), $\Xi_\infty := Z_\infty(I_n - \gamma^{-2}\bar{\Sigma}_\infty Z_\infty)^{-1} \in \mathcal{S}_{\mathrm{psd}\, n}$ is the stabilizing solution to the generalized algebraic Riccati equation (39), and the upper value of the game with cost function $J_{\gamma\infty}(\mu, \nu)$ is equal to 0. A controller guaranteeing this upper value is given by (40), which is the limiting controller of the finite-horizon case (23) with $\check{x}_0 = \mathbf{0}_n$, $Q_0 = \infty I_n$, and $Q_f = \mathbf{0}_{n\times n}$ as $t_0 \downarrow -\infty$ and $t_f \uparrow \infty$. Furthermore, the closed-loop system with (1) and (40) is asymptotically stable.*

Proof. By Assumption 5 and (ii) and (iii) of Proposition 3 and duality, the matrices $\bar{A}_{f1}$, $\bar{A}_f$, $\tilde{A}_f$, and $\tilde{A}_{f1}$ are Hurwitz, and $\lim_{t_0 \downarrow -\infty} \bar{\Sigma}(t; t_0) \uparrow \bar{\Sigma}_\infty$ and $\lim_{t_f \uparrow \infty} Z(t; t_f) \uparrow Z_\infty$, $\forall t \in \mathbb{R}$, the pair (A, B) is stabilizable, and the pair (A, C) is detectable.

By Theorem 4, we have $\Xi_\infty \in \mathcal{S}_{\mathrm{psd}\, n}$ is the stabilizing solution to (39), the controller (40) is stabilizing and achieves the upper value of the game, which is equal to zero. Clearly, the controller is the limiting controller of the finite-horizon case (23) with $\check{x}_0 = \mathbf{0}_n$, $Q_0 = \infty I_n$, $Q_f = \mathbf{0}_{n\times n}$ as $t_0 \downarrow -\infty$ and $t_f \uparrow \infty$.

This completes the proof of the corollary. □

8 An Example

We consider the ball and beam example. The nonlinear system dynamics is given by

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\beta x_2 + x_1 x_4^2 - G\sin(x_3) + 10 w_5 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= (1 + x_1^2)^{-1}(-G\cos(x_3)x_1 - 2x_1x_2x_4 + u + w_6) \\
y_1 &= x_1 + \frac{w_1}{1000} \\
y_2 &= x_2 + \frac{w_2}{100} \\
y_3 &= x_3 + \frac{w_3}{1000} \\
y_4 &= x_4 + \frac{w_4}{100} \\
z_1 &= \frac{x_1}{10} \\
z_2 &= x_2 \\
z_3 &= x_3 \\
z_4 &= 1000 x_4 \\
z_5 &= 10u
\end{aligned}$$

where x_1 is the displacement of the ball along the beam, x_3 is the deflection angle of the beam, u is the torque applied on the beam, β is the viscous friction paramenter, and G is the gravitational constant. The initial state for $\boldsymbol{x} = (x_1, \ldots, x_4)$ is $\boldsymbol{x}_0 \in \mathbb{R}^4$. $\boldsymbol{w} := (w_1, \ldots, w_6)$ is the disturbance input vector. The coefficients multiplying the measurement noises $(w_1, \ldots, w_4)$ reflect our view that we can measure the displacement and the deflection angle up to $\frac{1}{1000}$ error, but measure their derivatives up to $\frac{1}{100}$ error. w_5 is system disturbance that we assume to be multiplied by large gain 10, since the term $x_1 x_4^2$ has large destabilizing effect on the (x_1, x_2) dynamics, but will not appear in the linearized model. w_6 is the the system disturbance in the (x_3, x_4) dynamics, which is relatively easy to counter. So, we choose its coefficient to be 1. $\boldsymbol{z} = (z_1, \ldots, z_5)$ is the controlled output. We want the control law to pay little attention to x_1, and therefore the cost coefficient $\frac{1}{10}$. We want x_4 to remain small in magnitude, since the term $x_1 x_4^2$ has destabilizing effect and can't be controlled effectively, and therefore the cost coefficient 1000. The control effort should remain small, and therefore the cost coefficient 10.

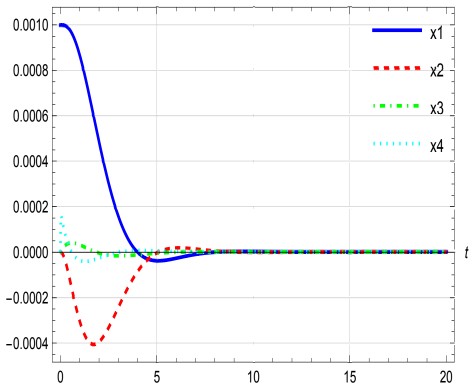
We linearize the system around $\bar{\boldsymbol{x}}_0 = \mathbf{0}_4$, $\bar{u}_0 = 0$, and $\bar{\boldsymbol{w}}_0 = \mathbf{0}_6$ to yield the linearized system:

$$
\begin{aligned}
\dot{\boldsymbol{x}}_l &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\beta & -G & 0 \\ 0 & 0 & 0 & 1 \\ -G & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x}_l + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_l + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{w}_l \\
\boldsymbol{y}_l &= I_4 \boldsymbol{x}_l + \begin{bmatrix} \frac{1}{1000} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{100} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1000} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{100} & 0 & 0 \end{bmatrix} \boldsymbol{w}_l \\
\boldsymbol{z}_l &= \begin{bmatrix} \frac{1}{10} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1000 \\ 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x}_l + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 10 \end{bmatrix} u_l
\end{aligned}
$$

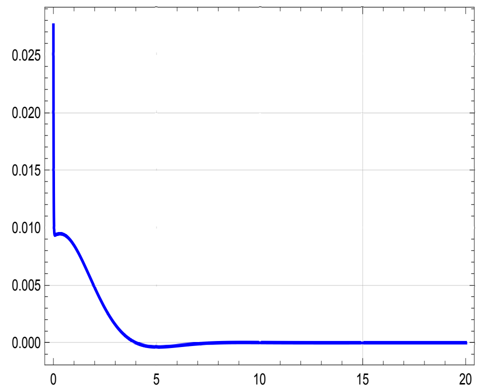
We set $\beta = \frac{1}{10}$ and $G = 9.81$.

This linearized system has $\gamma^* = 1301.8$. We choose $\gamma = 2000$ and design the H^∞ optimal controller.

$$
\begin{aligned}
\dot{\check{\boldsymbol{x}}}_l = & \begin{bmatrix} -9.9991 & 0.0099994 & 0.0041107 & 0.0089183 \\ -99.000 & -999.95 & -9.7135 & 0.016061 \\ 0.0041107 & 0.00096478 & -9.9587 & 0.090907 \\ 18.672 & 31.501 & -387.15 & -202.52 \end{bmatrix} \check{\boldsymbol{x}}_l \\
& + \begin{bmatrix} 9.9991 & 0.99000 & -0.0041107 & -0.0089185 \\ 99.000 & 999.85 & -0.096478 & -0.016061 \\ -0.0041107 & -0.00096478 & 9.9587 & 0.90912 \\ -0.89185 & -0.016061 & 90.912 & 99.588 \end{bmatrix} \boldsymbol{y}_l; \quad \check{\boldsymbol{x}}_l(0) = \check{\boldsymbol{x}}_{l0} \in \mathbb{R}^4
\end{aligned}
$$



(a)



(b)

Fig. 1 Closed loop performance of the H^∞ controller for $\gamma = 2000$, and $\boldsymbol{x}_0 = \check{\boldsymbol{x}}_{l0} = (0.001, 0, 0, 0)$ without any disturbances
(a) State trajectory; (b) Control input

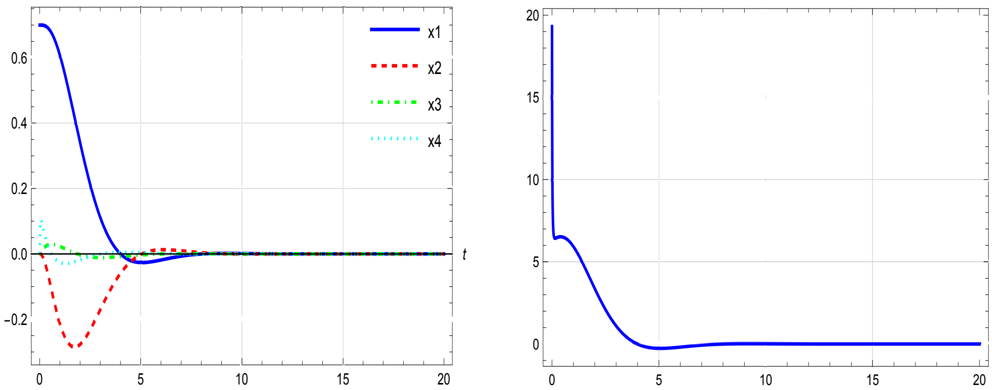
$$u_l = \begin{bmatrix} 27.590 & 31.485 & -296.24 & -102.94 \end{bmatrix} \check{\boldsymbol{x}}_l$$

The closed-loop system of the linearized system and the H^∞ controller is Hurwitz, with closed-loop eigenvalues -999.95, $-99.748+0.44766\text{i}$, $-99.748-0.44766\text{i}$, -10.052, -10.000, -1.6153, $-0.70808+0.76051\text{i}$, $-0.70808-0.76051\text{i}$.

Now, we apply the H^∞ controller on the ball and beam system (nonlinear) and simulate the system response when $\boldsymbol{w}(t) \equiv \boldsymbol{0}_6$, $\forall t \in \overline{\mathbb{R}_+}$. The initial condition for the system is set at $\boldsymbol{x}_0 = (0.001, 0, 0, 0) = \check{\boldsymbol{x}}_{l0}$. The system state trajectory is shown in Figure 1(a); and the control input is shown in Figure 1(b). We observe that the system is well behaved, the state trajectory is bounded in magnitude by 0.001, and the control input is bounded in magnitude by 0.028. The system admits multi-time-scale behavior. This simulation is to show the system performance when the system is very close to the equilibrium point.

Next, we set the initial condition for the system to $\boldsymbol{x}_0 = (0.7, 0, 0, 0) = \check{\boldsymbol{x}}_{l0}$. The system state trajectory is shown in Figure 2(a); and the control input is shown in Figure 2(b). We observe that the system is well behaved, the state trajectory is bounded in magnitude by 0.7, and the control input is bounded in magnitude by 19. The system response is very similar to the previous case. This simulation is to show the system performance when the system is away from the equilibrium point.

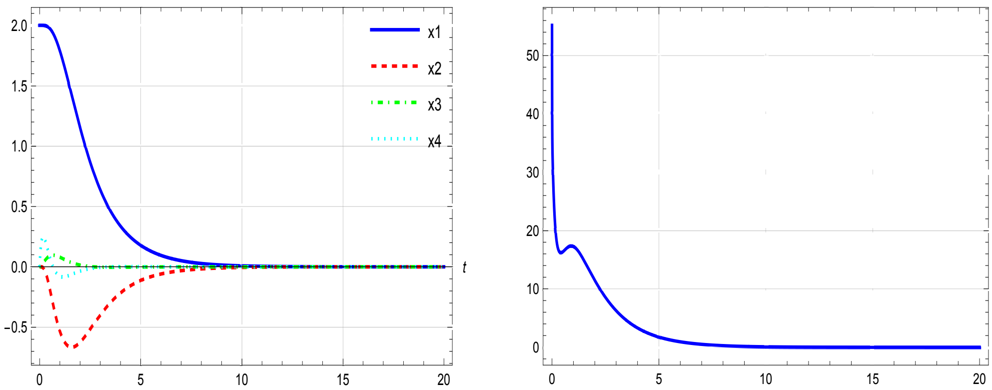
Next, we set the initial condition for the system to $\boldsymbol{x}_0 = (2.001, 0, 0, 0)$ and $\check{\boldsymbol{x}}_{l0} = (2, 0, 0, 0)$ and perturb the system parameters $\beta = 0.5$ and $G = 9.79$ in the ball and beam system. The system state trajectory is shown in Figure 3(a); and the control input is shown in Figure 3(b). We observe that the system is well behaved, the state trajectory is bounded in magnitude by 2, and the control input is bounded in magnitude by 56. The system response is similar to the previous cases despite of the initial state estimation error, and the system parameter uncertainty. This simulation is to show the system performance away from the equilibrium point and robustness to small perturbations.



(a) (b)

Fig. 2 Closed loop performance of the H^∞ controller for $\gamma = 2000$, and $\boldsymbol{x}_0 = \check{\boldsymbol{x}}_{l0} = (0.7, 0, 0, 0)$ without any disturbances

(a) State trajectory; (b) Control input



(a) (b)

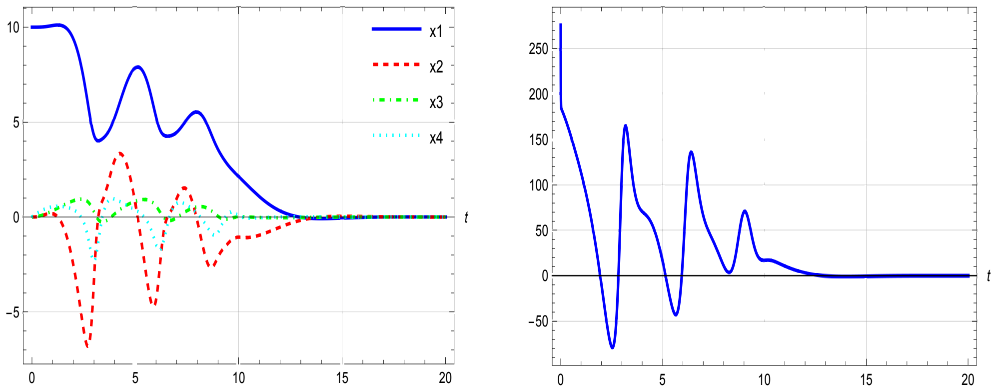
Fig. 3 Closed loop performance of the H^∞ controller for $\gamma = 2000$, $\boldsymbol{x}_0 = (2, 0, 0, 0)$, and $\check{\boldsymbol{x}}_{l0} = (2.001, 0, 0, 0)$ without any disturbances

(a) State trajectory; (b) Control input

Finally, we set the initial condition for the system to $\boldsymbol{x}_0 = \check{\boldsymbol{x}}_{l0} = (10, 0, 0, 0)$. The system state trajectory is shown in Figure 4(a); and the control input is shown in Figure 4(b). We observe that the state trajectory is bounded in magnitude by 11 and the control input is bounded in magnitude by 280. The system response is very oscillatory but eventually converges to the origin. This simulation is to show the system performance near the boundary of the region of attraction for the linear controller.

9 Conclusion

In this paper, I have studied both the finite-horizon and the infinite-horizon H^∞-optimal control problem under imperfect state measurements using a game theoretic



(a) (b)

Fig. 4 Closed loop performance of the H^∞ controller for $\gamma = 2000$, and $\boldsymbol{x}_0 = \check{\boldsymbol{x}}_{l0} = (10, 0, 0, 0)$ without any disturbances

(a) State trajectory; (b) Control input

approach. The problem is converted into a soft-constrained zero-sum differential game. The key is to find the upper value of the game, and find the minimizer policy that guarantees the finite upper value for the game. In the finite-horizon case, I solve the upper value of the game in an estimator and controller sequential design fashion, where the Riccati differential equation obtained for the control step is something a bit unfamiliar. I solve this Riccati differential equation explicitly and recover the well-known solution to the problem. Then, I study the infinite-horizon H^∞-optimal control problem under imperfect state measurements. I present a significant result that for a generalized algebraic Riccati equation $\bar{A}'Z + Z\bar{A} - Z\bar{S}Z + \bar{Q} = \mathbf{0}$ to admit a positive-semidefinite stabilizing solution, then necessarily the pair $(\bar{A}, \bar{Q})$ has no unobservable mode on the imaginary axis. I prescribe a set of exact conditions under which the system under the optimal controller is internally stable and achieves the 0 upper value for the game. This result, together with Theorem 9.8 of Başar and Bernhard (1995), then completely solves the infinite-horizon H^∞-optimal control problem under imperfect state measurements. I prescribe another set of exact conditions under which the optimal control strategy for the infinite-horizon case is the appropriate limit of the optimal control strategy for the finite-horizon case as the time interval expands to the entire real line, the optimal controller guarantees the 0 upper value for the game, and such that the system under the optimal controller is internally stable. This set of conditions is standard in the literature. This result validates the receding horizon H^∞-optimal control design methodology.

Due to the equivalence of H^∞ central controller with the optimal controller for LEQG problem, the result in this paper has direct ramifications in LEQG problem.

Future research directions are to investigate LEQG problem, study the time-varying systems, study the H^∞ control and/or LEQG control for singularly perturbed systems, investigate possible generalization to jump linear systems, and study extension to nonlinear systems.

Acknowledgements. The author was supported by National Science Foundation CAREER Grant ECS-9702702 and National Science Foundation Grant ECS-0296071; and by National Natural Science Foundation of China Grant 60004005.

References

Başar, T., Bernhard, P.: H^∞-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach, 2nd edn. Birkhäuser, Boston, MA (1995)

Bensoussan, A., Elliott, R.J.: General finite-dimensional risk-sensitive problems and small noise limits. IEEE Transactions on Automatic Control **41**(2), 210–215 (1996)

Bernstein, D.S., Haddad, W.M.: LQG control with an H_∞ performance bound: a Riccati equation approach. IEEE Transactions on Automatic Control **34**(3), 293–305 (1989)

Bamieh, B., Pearson, J.B.: A general framework for linear periodic systems with application to H^∞ sampled-data control. IEEE Transactions on Automatic Control **37**, 418–435 (1992)

Didinsky, G., Başar, T.: Design of minimax controllers for linear systems with nonzero initial states under specified information structures. International Journal of Robust and Nonlinear Control **2**, 1–30 (1992)

Didinsky, G., Başar, T., Bernhard, P.: Structural properties of minimax controllers for a class of differential games arising in nonlinear H^∞-control. Systems & Control Letters **21**, 433–441 (1993)

de Souza, C.E., Fragoso, M.: H^∞ control of linear systems with Markovian jumping parameters. Control Theory and Technology **9**(2), 457–466 (1993)

Doyle, J.C., Francis, B.A., Tannenbaum, A.R.: Feedback Control Theory. Macmillan, New York, NY (1992)

Doyle, J., Glover, K., Khargonekar, P.P., Francis, B.: State-space solutions to standard H_2 and H_∞ control problems. IEEE Transactions on Automatic Control **34**(8), 831–847 (1989)

Fleming, W.H., McEneaney, W.M.: Risk sensitive control with ergodic cost criteria. In: Proceedings of the 31st Conference on Decision and Control, Tucson, AZ, pp. 2048–2052 (1992)

Fleming, W.H., McEneaney, W.M.: Risk-sensitive control on an infinite time horizon. SIAM J. Control and Optimization **33**, 1881–1915 (1995)

Isidori, A., Astolfi, A.: Disturbance attenuation and H_∞-control via measurement feedback in nonlinear systems. IEEE Transactions on Automatic Control **37**, 1283–1293 (1992)

James, M.R., Baras, J.S., Elliott, R.J.: Output feedback risk-sensitive control and differential games for continuous-time nonlinear systems. In: Proceedings of the 32nd IEEE Conference on Decision and Control, San Antonio, TX, pp. 3357–3360 (1993)

James, M.R., Yuliar, S.: A nonlinear partially observed differential game with a finite-dimensional information state. Systems & Control Letters **26**, 137–145 (1995)

Khargonekar, P.P., Nagpal, K.N., Poolla, K.R.: H^∞ control with transients. SIAM Journal on Control and Optimization **29**, 1373–1393 (1991)

Khargonekar, P.P., Rotea, M.A.: Mixed H_2/H_∞ control: a convex optimization approach. IEEE Transactions on Automatic Control **36**(7), 824–837 (1991)

Limebeer, D., Anderson, B.D.O., Khargonekar, P.P., Green, M.: A game theoretic approach to H_∞ control for time varying systems. In: Proceedings of the International Symposium on the Mathematical Theory of Networks and Systems, Amsterdam, The Netherlands (1989)

Moon, J., Duncan, T.E., Başar, T.: Risk-sensitive zero-sum differential games. IEEE Transactions on Automatic Control **64**(4), 1503–1518 (2019) https://doi.org/10.1109/TAC.2018.2846048

Pan, Z.: Measure-theoretic Calculus in Abstract Spaces: On the Playground of Infinite-dimensional Spaces, 1st edn. Birkhäuser, Cham, Switzerland (2024). https://doi.org/10.1007/978-3-031-21912-2

Pan, Z., Başar, T.: H^∞-optimal control for singularly perturbed systems. Part II: Imperfect state measurements. IEEE Transactions on Automatic Control **39**(2), 280–299 (1994)

Pan, Z., Başar, T.: Model simplification and optimal control of stochastic singularly perturbed systems under exponentiated quadratic cost. SIAM J. Control and Optimization **34**(5), 1734–1766 (1996)

Pan, Z., Başar, T.: Random evolutionary time-scale decomposition in robust control of jump linear systems. In: Proceedings of the 35th IEEE Conference on Decision and Control, Kobe, Japan, pp. 517–522 (1996)

Pan, Z., Başar, T.: Adaptive controller design and disturbance attenuation for SISO linear systems with noisy output measurements. CSL report, University of Illinois at Urbana-Champaign, Urbana, IL (July 2000). For an extended version of that report: https://tinyurl.com/y4zztevj

Ravi, R., Nagpal, K.M., Khargonekar, P.P.: H_∞ control for linear time varying systems: a state-space approach. SIAM Journal on Control and Optimization **29**, 1394–1413 (1991)

Rhee, I., Speyer, J.L.: A game theoretic controller and its relationship to H^∞ and linear-exponential Gaussian synthesis. In: Proceedings of the 28th IEEE Conference on Decision and Control, Tampa, FL, pp. 909–915 (1989)

Runolfsson, T.: The equivalence between infinite horizon control of stochastic systems with exponential-of-integral performance index and stochastic differential games. IEEE Transactions on Automatic Control **39**, 1551–1563 (1994)

Stoorvogel, A.A.: The H_∞ Control Problem: A State Space Approach. Prentice Hall, Englewood Cliffs (1992)

Stoorvogel, A.A.: The robust H_2 control problem: a worst-case design. IEEE Transactions on Automatic Control **38**(9), 1358–1370 (1993)

Toussaint, G.J., Başar, T., Bullo, F.: H^∞-optimal tracking control techniques for nonlinear underactuated systems. In: Proceedings of the 39th IEEE Conference on Decision and Control, vol. 3. Sydney, Australia, pp. 2078–2083 (2000). https://doi.org/10.1109/CDC.2000.914100

Trentelman, H.L., Stoorvogel, A.A., Hautus, M.: Control Theory for Linear Systems. Springer, London (2001)

Uchida, K., Fujita, M.: On the central controller: Characterizations via differential games and LEQG control problems. Systems and Control Letters **13**(1), 9–13 (1989)

van der Schaft, A.J.: On a state-space approach to nonlinear H_∞ control. Systems & Control Letters **16**, 1–8 (1991)

Whittle, P.: Risk-Sensitive Optimal Control. John Wiley and Sons, Chichester, NY (1990)

Zames, G.: Feedback and optimal sensitivity: Model reference transformation, multiplicative seminorms and approximate inverses. IEEE Transactions on Automatic Control **26**, 301–320 (1981)

Zhou, K., Doyle, J., Glover, K.: Robust and Optimal Control. Prentice Hall, Upper Saddle River, NJ (1996)