Properties of the Generalized Minimum Phase Concept for MIMO LTI Systems with Additive Disturbances†

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***Abstract*— In [1], we have introduced a generalized concept of minimum phase for finite-dimensional continuous-time multiple-input and multiple-output linear time-invariant systems with additive disturbances. In this paper, we investigate further properties of minimum phase systems using this concept. We prove that a minimum phase system with uniform vector relative degree (UVRD) in feedback connection with another linear system satisfying a certain boundedness condition yields a minimum phase composite system. We establish that a minimum phase system with UVRD may be inverted, that is an arbitrary reference signal with bounded derivatives up to certain order can be tracked, without making the internal states unbounded. When a minimum phase linear system has UVRD *zero* from the control input to the output, has a bounded admissible initial condition, and a bounded admissible disturbance waveform, then the control input and the state trajectory are bounded if the output of the system is bounded. When a minimum phase linear system has UVRD $r_1 \in \mathbb{N}$, has a bounded admissible initial condition, and a bounded admissible disturbance waveform, if the noiseless output together with its noiseless derivatives up to k_0th ($0 \leq k_0 < r_1$) order are bounded, then the output of a "stable" system sharing the same inputs as the minimum phase system is bounded if the relative degrees of the outputs of the "stable" system satisfy $r_2 \geq r_1 - k_0$. These results have significant implications on model reference control theory.**

***Index Terms*— continuous-time systems, extended zero dynamics canonical form, minimum phase, extended zero dynamics.**

I. Introduction

The minimum phase property is of paramount importance in model reference control theory, attracting sustained research attention [2], [3], [4], [5], [6]. In an earlier paper [1], we generalized the minimum phase concept for multiple-input and multiple-output (MIMO) LTI systems with additive disturbance inputs in an attempt to make it necessary for solvability of the output feedback model reference control problem. When it is right invertible, then the system can be dynamically extended to admit uniform vector relative degree (UVRD). For this extended system, it may be transformed into the extended zero dynamics canonical form (EZDCF) representation. Based on this canonical form representation, the extended zero dynamics (EZD) for the system can be readout, which is the zero dynamics for the original system and is simply the zero dynamics as defined in [3] together with the driving terms, including the output and the disturbance input of the system. The original system is said to be minimum phase with respect to the given set of admissible initial conditions and the given set of admissible disturbance waveforms if the EZD is absent or satisfies the properties that the zero dynamics state is bounded for any bounded admissible initial condition (for the entire system), any bounded output waveform, and any bounded admissible disturbance waveform. In our earlier paper, the relationship of the generalized concept of minimum phase with that introduced in [3] and [2] was investigated.

In this paper, we investigate further properties of minimum phase systems using the definition in [1]. We prove that a composite system consisting of a minimum phase system in feedback connection with another linear system satisfying a certain boundedness condition is itself a minimum phase system. We further establish the inversion result for square minimum phase systems as defined in [1]. When a square minimum phase linear system has the UVRD $r_1 = 0$ from the control input to the output, has a bounded admissible initial condition, and a bounded admissible disturbance waveform, then the control input and the state trajectory are bounded if the output of the system is bounded. When a square minimum phase linear system has UVRD $r_1 \in \mathbb{N}$, has a bounded admissible initial condition, and a bounded admissible disturbance waveform, if the noiseless output together with its noiseless derivatives up to k_0th ($0 \leq k_0 < r_1$) order are bounded, then the output of a "stable" linear system sharing the same inputs as the minimum phase system is bounded if the relative degrees of the outputs of the "stable" system satisfy $r_2 \geq r_1 - k_0$. These results have significant implications on model reference control theory.

The balance of the paper is as follows. In the next section, we list the notations used in the paper. Then, in Section III, the minimum phase property of the composite system is proved for feedback interconnected systems. The boundedness of the inverse of minimum phase systems is presented in Section IV. The paper ends with some concluding remarks in Section V, and an appendix containing some useful technical lemmas necessary for the derivations in the main body of the paper.

II. Notations

Let $\mathbb{R}$ denote the real line; $\mathbb{R}_e := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$; $\mathbb{N}$ be the set of natural numbers; $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Unless specified, all signals, constants, and matrices are real. For a continuous function f, we say that it belongs to $\mathcal{C}$. We say that a function is L_∞ if it is bounded. For any matrix A, A' denotes its transpose. For any $z \in \mathbb{R}^n$, $|z|$ denotes $\sqrt{z'z}$. I_n denotes the $n \times n$-dimensional identity matrix. For any

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matrix A, $A^0 = I$. For any matrix M, $\|M\|$ denotes its 2-induced norm. $\mathbf{0}_{m\times n}$ denotes the $m\times n$-dimensional matrix whose elements are all zeros. For any waveform $u_{[0,t_f)} \in \mathcal{C}([0,t_f), \mathbb{R}^p)$, where $t_f \in (0,\infty] \subset \mathbb{R}_e$ and $p \in \mathbb{Z}_+$, $\|u_{[0,t_f)}\|_\infty = \sup_{t\in[0,t_f)} |u(t)|$.

III. Minimum phase property for feedback interconnected systems

We first present a lemma which establishes the minimum phase property for MIMO LTI systems in a general canonical form, which arises in interconnected systems.

Lemma 1: Consider a MIMO LTI system

$$\dot{x}_z = A_o x_z + A_{o1}x_1 + \cdots + A_{or}x_r + A_{o\,r+1}b_0 u + D_o w \quad (1a)$$
$$\dot{x}_i = A_{io}x_z + A_{i1}x_1 + \cdots + A_{ii}x_i + x_{i+1} + D_i w; \quad 1\le i<r \quad (1b)$$
$$\dot{x}_r = A_{ro}x_z + A_{r1}x_1 + \cdots + A_{rr}x_r + B_0 u + D_r w \quad (1c)$$
$$y = x_1 + Ew \quad (1d)$$

where $x_z \in \mathbb{R}^{n_z}$, $n_z \in \mathbb{N}$; $x_i \in \mathbb{R}^m$, $i = 1,\ldots,r$, $r \in \mathbb{N}$; $B_0 \in \mathbb{R}^{m\times m}$ is invertible; $y \in \mathbb{R}^m$ is the output, $m \in \mathbb{N}$; $u \in \mathbb{R}^m$ is the control input; $w \in \mathbb{R}^q$ is the disturbance input, $q \in \mathbb{Z}_+$; all matrices are of appropriate dimensions and constant. Let $x = [x_z'\, x_1' \cdots x_r']'$, $x(0) = x_0 \in \mathcal{D}_0 \subseteq \mathbb{R}^n$, $\mathcal{D}_0$ is a subspace, $n = n_z + mr \in \mathbb{N}$; $w_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_q$ (See Definition 2 of [5]). Assume that $\exists m_o \in \{1,\ldots,r+1\}$ such that $A_{oj} = \mathbf{0}_{n_z\times m}$, $\forall j \in \{m_o+1,\ldots,r+1\}$; and $A_{jo} = \mathbf{0}_{m\times n_z}$, $\forall j \in \{1,\ldots,m_o-1\}$. Then, system (1) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ and admits UVRD r from u to y if the dynamics (1a) satisfies $\forall c_w \ge 0$, $\exists c_c \ge 0$, $\forall x_0 \in \mathcal{D}_0$ with $|x_0| \le c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \le c_w$, $\forall x_{i[0,\infty)} \in \mathcal{C}$ with $\|x_{i[0,\infty)}\|_\infty \le c_w$, $i = 1,\ldots,m_o$, ($x_{r+1} := b_0 u$ for notational consistency) we have $\|x_{z[0,\infty)}\|_\infty \le c_c$.

Proof: We will prove the lemma using mathematical induction on m_o.

1° $m_o = 1$. Clearly $A_{o2} = \cdots = A_{o\,r+1} = \mathbf{0}_{n_z\times m}$. Then, it is easy to show that (1) admits UVRD r, and furthermore, by Proposition 2 of [1], (1a) is the EZD of (1) according to Definition 1 of [1]. Then, (1) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. This proves 1°.

2° Suppose that the lemma holds for $m_o \le k (\le r)$.

3° Consider the case $m_o = k+1 \in \{2,\ldots,r+1\}$. Consider the state transformation $\bar{x} := [\bar{x}_z'\, x_1' \cdots x_r']' = T_1^{-1}x = \left[x_z' - x_k' A_{o\,k+1}'\, x_1' \cdots x_r'\right]'$. Then, in $\bar{x}$ coordinates, system (1) admits the representation with $\bar{x}(0) = \bar{x}_0 := \left[\bar{x}_{z0}'\, x_{1,0}' \cdots x_{r,0}'\right]' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_0)$, which is a subspace of $\mathbb{R}^n$,

$$\begin{aligned}\dot{\bar{x}}_z &= A_o\bar{x}_z + A_oA_{o\,k+1}x_k + A_{o1}x_1 + \cdots + A_{ok}x_k \\ &\quad + D_o w - A_{o\,k+1}\left(A_{k1}x_1 + \cdots + A_{kk}x_k + D_k w\right) \\ &=: A_o\bar{x}_z + \bar{A}_{o1}x_1 + \cdots + \bar{A}_{ok}x_k + \bar{D}_o w \end{aligned} \quad (2a)$$
$$\dot{x}_i = A_{i1}x_1 + \cdots + A_{ii}x_i + x_{i+1} + D_i w; \quad i = 1,\ldots,k \quad (2b)$$
$$\begin{aligned}\dot{x}_i &= A_{io}\bar{x}_z + A_{io}A_{o\,k+1}x_k + A_{i1}x_1 + \cdots + A_{ii}x_i \\ &\quad + x_{i+1} + D_i w; \quad i = k+1,\ldots,r \end{aligned} \quad (2c)$$
$$y = x_1 + Ew \quad (2d)$$

Clearly, the representation (2) is in the form of (1) with $\bar{m}_o = k$. We will apply the inductive argument to show that (2) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. Toward this end, $\forall c_w \ge 0$, $\forall \bar{x}_0 \in \bar{\mathcal{D}}_0$ with $|\bar{x}_0| \le c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \le c_w$, $\forall x_{i[0,\infty)} \in \mathcal{C}$ with $\|x_{i[0,\infty)}\|_\infty \le c_w$, $i = 1,\ldots,k$. Since $\bar{x}_0 \in \bar{\mathcal{D}}_0 = T_1^{-1}(\mathcal{D}_0)$, then $\acute{x}_0 := [\acute{x}_{z0}'\, \acute{x}_{1,0} \cdots \acute{x}_{r,0}] := T_1\bar{x}_0 \in \mathcal{D}_0$ with $|\acute{x}_0| \le \|T_1\|c_w$. The solution to (2a) may be decomposed by linearity as the sum of the following three systems.

$$\begin{aligned}\dot{\eta}_1 &= A_o\eta_1 + A_{o1}x_1 + \cdots + A_{ok}x_k + A_{o\,k+1}(-A_{k1}x_1 - \cdots \\ &\quad - A_{kk}x_k - D_k w) + D_o w; \; \eta_1(0) = \acute{x}_{z0} \end{aligned} \quad (3a)$$
$$\dot{\eta}_2 = A_o\eta_2; \qquad \eta_2(0) = -A_{o\,k+1}\acute{x}_{k,0} \quad (3b)$$
$$\dot{\eta}_3 = A_o\eta_3 + A_oA_{o\,k+1}x_k; \qquad \eta_3(0) = \mathbf{0}_{(n-r)\times 1} \quad (3c)$$
$$\bar{x}_{z[0,\infty)} = \eta_{1[0,\infty)} + \eta_{2[0,\infty)} + \eta_{3[0,\infty)} \quad (3d)$$

Let $\bar{x}_{k+1} = -A_{k1}x_1 - \cdots - A_{kk}x_k - D_k w$. Then, $\bar{x}_{k+1[0,\infty)} \in \mathcal{C}$ and $\|\bar{x}_{k+1[0,\infty)}\|_\infty \le (\sum_{i=1}^k \|A_{ki}\|)c_w + \|D_k\|c_w =: \bar{c}_{w1}c_w$. For (3a), by the assumption of the lemma, $\exists c_{c1} \ge 0$, which depends only on $\|T_1\|c_w$, $\bar{c}_{w1}c_w$, and c_w, such that $\|\eta_{1[0,\infty)}\|_\infty \le c_{c1}$.

Under the assumption of the lemma, by Lemma 9 of [5], the following system is bounded input bounded state (BIBS) stable: $\dot{\xi} = A_o\xi + A_{o\,k+1}v$, $\xi(0) = \mathbf{0}_{(n-r)\times 1}$. For (3b), by Lemma 3, $\exists c_{c2} \ge 0$, which does not depend on any other constant, such that $\|\eta_{2[0,\infty)}\|_\infty \le c_{c2}\|A_{o\,k+1}\||\acute{x}_{k,0}| \le c_{c2}\|A_{o\,k+1}\|c_w$.

Again by Lemma 3, (3c) is BIBS stable. Then, by Lemma 6 of [5], $\exists c_{c3} \ge 0$, which does not depend on any other constant, such that $\|\eta_{3[0,\infty)}\|_\infty \le c_{c3}c_w$.

Then, we have $\|\bar{x}_{z[0,\infty)}\|_\infty \le c_{c1} + c_{c2}\|A_{o\,k+1}\|c_w + c_{c3}c_w$. This shows that (2) satisfies the inductive assumption. Hence, (2) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ and admits UVRD r, which is equivalent to that (1) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$ and admits UVRD r from u to y.

This completes the induction process and the proof. ■

Next, we present a theorem that establishes the minimum phase property for the composite system consisting of two subsystems in feedback configuration. The block diagram of the system is shown in Figure 1.

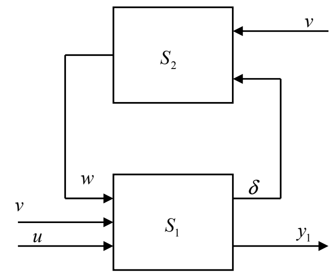

Fig. 1. Block diagram of two feedback interconnected systems.

Theorem 1: Consider two LTI systems in feedback:

$$S_1 : \begin{cases} \dot{x} = A_1x + B_1u + D_{1,w}w + D_{1,v}v; \; x(0) = x_0 \\ y = C_1x + K_1u + E_{1,w}w + E_{1,v}v \\ \delta = C_1x + K_1u \end{cases} \quad (4a)$$
$$S_2 : \begin{cases} \dot{\eta} = A_2\eta + B_2\delta + D_2v; \qquad \eta(0) = \eta_0 \\ w = C_2\eta + K_2\delta + E_2v \end{cases} \quad (4b)$$

where $x \in \mathbb{R}^{n_1}$ is the state for S_1, $n_1 \in \mathbb{Z}_+$; $y \in \mathbb{R}^{m_1}$ is the output of S_1, $m_1 \in \mathbb{N}$; $u \in \mathbb{R}^{m_1}$ is the input for S_1; $\delta \in \mathbb{R}^{m_1}$ is the noiseless output of S_1, which is also the input to S_2; $w \in \mathbb{R}^{q_w}$ is the disturbance input for S_1, $q_w \in \mathbb{Z}_+$, which is also the output of S_2; $v \in \mathbb{R}^{q_v}$ is the disturbance input for both S_1 and S_2, $q_v \in \mathbb{Z}_+$; $\eta \in \mathbb{R}^{n_2}$ is the state for S_2, $n_2 \in \mathbb{Z}_+$; $x_0 \in \mathcal{D}_{x0}$, $\mathcal{D}_{x0} \subseteq \mathbb{R}^{n_1}$ is a subspace; $\eta_0 \in \mathcal{D}_{\eta 0}$, $\mathcal{D}_{\eta 0} \subseteq \mathbb{R}^{n_2}$ is a subspace; $v_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_{q_v}$; and all matrices are constant and of appropriate

TABLE I

6 EXHAUSTIVE AND MUTUALLY EXCLUSIVE CASES FOR THEOREM 1.

Case 1:	$0 = r_1 < n_1/m_1$ and $n_2 \in \mathbb{Z}_+$
Case 2:	$1 = r_1 = n_1/m_1$ and $n_2 \in \mathbb{Z}_+$
Case 3:	$2 \le r_1 = n_1/m_1$ and $n_2 \in \mathbb{Z}_+$
Case 4:	$1 = r_1 < n_1/m_1$ and $n_2 \in \mathbb{Z}_+$
Case 5:	$2 \le r_1 < n_1/m_1$ and $n_2 \in \mathbb{Z}_+$
Case 6:	$0 = r_1 = n_1/m_1$ and $n_2 \in \mathbb{Z}_+$

dimensions.

The composite system S has control input u, output y, and disturbance v. Assume that

1) S_1 admits UVRD $r_1 \in \{0, \ldots, \lfloor \frac{n_1}{m_1} \rfloor\}$ from u to y and is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{C}([0,\infty), \mathbb{R}^{q_w}) \times \mathcal{W}_d$, where w and v are considered as disturbances;
2) $\hat{E}_{1,w} := I_{m_1} + E_{1,w}K_2$ is invertible, such that S admits a well defined UVRD; and
3) the following associated system of S_2, $\bar{S}_2$,
$$\begin{cases} \dot{\psi} = (A_2 - B_2\hat{E}_{1,w}^{-1}E_{1,w}C_2)\psi + B_2\tau + D_2 v \\ \quad =: \bar{A}_2\psi + B_2\tau + D_2 v; \quad \psi(0) = \eta_0 \in \mathcal{D}_{\eta 0} \\ \phi = \tau \end{cases} \tag{5}$$
with control input τ and disturbance v and output $\phi \in \mathbb{R}^{m_1}$ is minimum phase with respect to $\mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$.

Then, the composite system S admits UVRD r_1 from u to y and is minimum phase with respect to $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$.

Proof: Note that
$$\begin{aligned} y &= \delta + E_{1,w}(C_2\eta + K_2\delta + E_2 v) + E_{1,v}v \\ &= (I_{m_1} + E_{1,w}K_2)\delta + E_{1,w}C_2\eta + (E_{1,v} + E_{1,w}E_2)v \\ &=: \hat{E}_{1,w}\delta + E_{1,w}C_2\eta + \hat{E}_{1,v}v \end{aligned} \tag{6}$$

We will distinguish 6 exhaustive and mutually exclusive cases, which are listed in Table I.

Case 1: $0 = r_1 < \frac{n_1}{m_1}$ and $n_2 \in \mathbb{Z}_+$. By Lemma 4 of [1], without loss of generality, assume that S_1 is given in the following EZDCF representation, K_1 being invertible,
$$\begin{aligned} \dot{x} &= \bar{A}_1 x + \bar{B}_1(y - E_{1,w}w - E_{1,v}v) + D_{1,w}w + D_{1,v}v \\ &=: \bar{A}_1 x + \bar{B}_1 y + \bar{D}_{1,w}w + \bar{D}_{1,v}v; \qquad x(0) = x_0 \\ y &= \delta + E_{1,w}w + E_{1,v}v \\ \delta &= C_1 x + K_1 u \end{aligned}$$
Note that, by (6),
$$\begin{aligned} \delta &= \hat{E}_{1,w}^{-1}(y - E_{1,w}C_2\eta - \hat{E}_{1,v}v) \\ w &= C_2\eta + K_2\delta + E_2 v =: \bar{C}_2\eta + K_2\hat{E}_{1,w}^{-1}y + \bar{E}_2 v \end{aligned}$$
Then, the composite system S admits the following representation, in $\xi = [x' \, \eta']'$ coordinates,
$$\begin{aligned} \dot{x} &= \bar{A}_1 x + \bar{B}_1 y + \bar{D}_{1,w}(\bar{C}_2\eta + K_2\hat{E}_{1,w}^{-1}y + \bar{E}_2 v) + \bar{D}_{1,v}v \\ &=: \bar{A}_1 x + \hat{A}_{12}\eta + \hat{B}_1(y - \hat{E}_{1,v}v) + \hat{D}_{1,v}v \end{aligned} \tag{7a}$$
$$\begin{aligned} \dot{\eta} &= A_2\eta + B_2\hat{E}_{1,w}^{-1}(y - E_{1,w}C_2\eta + \hat{E}_{1,v}v) + D_2 v \\ &= \bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}(y - \hat{E}_{1,v}v) + D_2 v \end{aligned} \tag{7b}$$
$$y = \hat{E}_{1,w}C_1 x + E_{1,w}C_2\eta + \hat{E}_{1,w}K_1 u + \hat{E}_{1,v}v \tag{7c}$$
Clearly, (7) is in the EZDCF (14) of [1] with (7a) and (7b) as its EZD, since $\hat{E}_{1,w}K_1$ is invertible. Hence, S admits UVRD $0 = r_1$ from u to y. $\forall c_w \ge 0$, $\forall [x_0' \, \eta_0']' \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ with $|[x_0' \, \eta_0']'| \le c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \le c_w$, $\forall y_{[0,\infty)} \in \mathcal{C}$ with $\|y_{[0,\infty)}\|_\infty \le c_w$, let $x_{[0,\infty)}$ and $\eta_{[0,\infty)}$ be the solution to (7a) and (7b). Let $\tau = \hat{E}_{1,w}^{-1}(y - \hat{E}_{1,v}v)$. Then, $\tau_{[0,\infty)} \in \mathcal{C}$ with $\|\tau_{[0,\infty)}\|_\infty \le (\|\hat{E}_{1,w}^{-1}\| + \|\hat{E}_{1,w}^{-1}\hat{E}_{1,v}\|)c_w =: c_{w1}$. By the assumption on the system $\bar{S}_2$, $\exists c_{c1} \ge 0$, which depends only on c_w and c_{w1}, such that $\|\eta_{[0,\infty)}\|_\infty \le c_{c1}$. Note that $w_{[0,\infty)} \in \mathcal{C}$ with $\|w_{[0,\infty)}\|_\infty \le \|\bar{C}_2\|c_{c1} + \|K_2\hat{E}_{1,w}^{-1}\|c_w + \|\bar{E}_2\|c_w =: c_{w2}$. Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{C}([0,\infty), \mathbb{R}^{q_w}) \times \mathcal{W}_d$, $\exists c_{c2} \ge 0$, which depends only of c_w and c_{w2}, such that $\|x_{[0,\infty)}\|_\infty \le c_{c2}$. Then, $\|\xi_{[0,\infty)}\|_\infty \le \sqrt{c_{c1}^2 + c_{c2}^2}$. Hence, (7) is minimum phase with respect to $\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$. This proves Case 1.

Case 2: $1 = r_1 = \frac{n_1}{m_1}$ and $n_2 \in \mathbb{Z}_+$. By Lemma 3 of [1], without loss of generality, assume that S_1 is given in the following EZDCF representation, B_1 being invertible,
$$\begin{aligned} \dot{x}_1 &= A_{1,11}x_1 + B_1 u + D_{1,1w}w + D_{1,1v}v \\ \delta &= x_1; \quad y = x_1 + E_{1,w}w + E_{1,v}v \end{aligned}$$
where $x = x_1 \in \mathbb{R}^{m_1}$; $x(0) = x_{1,0} \in \mathcal{D}_{x0} \subseteq \mathbb{R}^{m_1}$. Note that, by (6),
$$y = \hat{E}_{1,w}x_1 + E_{1,w}C_2\eta + \hat{E}_{1,v}v =: \bar{x}_1 + \hat{E}_{1,v}v$$
The system S admits the following representation, in $\xi = [\eta' \, \bar{x}_1']' = T_1^{-1}[x' \, \eta']'$ coordinates with $\xi(0) = \xi_0 := [\eta_0' \, \bar{x}_{1,0}']' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0})$,
$$\begin{aligned} \dot{\eta} &= A_2\eta + B_2 x_1 + D_2 v = \bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}\bar{x}_1 + D_2 v \\ &= \bar{A}_2\eta + \bar{B}_2\bar{x}_1 + D_2 v \end{aligned} \tag{8a}$$
$$\begin{aligned} \dot{\bar{x}}_1 &= E_{1,w}C_2(\bar{A}_2\eta + \bar{B}_2\bar{x}_1 + D_2 v) + \hat{E}_{1,w}(A_{1,11}x_1 \\ &\quad + B_1 u + D_{1,1w}(C_2\eta + K_2 x_1 + E_2 v) + D_{1,1v}v) \\ &=: \bar{A}_{1,1\eta}\eta + \bar{A}_{1,11}\bar{x}_1 + \hat{E}_{1,w}B_1 u + \bar{D}_{1,1v}v \end{aligned} \tag{8b}$$
$$y = \bar{x}_1 + \hat{E}_{1,v}v \tag{8c}$$
We will further distinguish two exhaustive and mutually exclusive subcases: Case 2a: $n_2 \in \mathbb{N}$; Case 2b: $n_2 = 0$.

Case 2a: $n_2 \in \mathbb{N}$. Clearly, (8) is in EZDCF (11) of [1], and admits UVRD $1 = r_1$ from u to y, since $\hat{E}_{1,w}B_1$ is invertible. Note that (8) is in the form of (1) with $m_o = 1$. $\forall c_w \ge 0$, $\forall (\eta_0, \bar{x}_{1,0}) \in \bar{\mathcal{D}}_0$ with $|(\eta_0, \bar{x}_{1,0})| \le c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \le c_w$, $\forall \bar{x}_{1[0,\infty)} \in \mathcal{C}$ with $\|\bar{x}_{1[0,\infty)}\|_\infty \le c_w$, let $\eta_{[0,\infty)}$ be the solution to (8a). We have $\eta_0 \in \mathcal{D}_{\eta 0}$ and $|\eta_0| \le c_w$. Let $\tau = \hat{E}_{1,w}^{-1}\bar{x}_1$. Then, $\tau_{[0,\infty)} \in \mathcal{C}$ with $\|\tau_{[0,\infty)}\|_\infty \le \|\hat{E}_{1,w}^{-1}\|c_w =: c_{w1}$. By the assumption on the system $\bar{S}_2$, $\exists c_{c1} \ge 0$, which depends only on c_w and c_{w1}, such that $\|\eta_{[0,\infty)}\|_\infty \le c_{c1}$. Hence, (8) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ by Lemma 1. This completes the proof for Case 2a.

Case 2b: $n_2 = 0$. (8) is in the EZDCF (13) of [1]. Then, (8) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ since its EZD is absent. This proves Case 2b and Case 2.

Case 3: $2 \le r_1 = \frac{n_1}{m_1}$ and $n_2 \in \mathbb{Z}_+$. By Lemma 3 of [1], without loss of generality, assume that S_1 is given in the following EZDCF representation, B_1 being invertible,
$$\begin{aligned} \dot{x}_i &= A_{1,i1}x_1 + x_{i+1} + D_{1,iw}w + D_{1,iv}v; \; 1 \le i < r_1 \\ \dot{x}_{r_1} &= A_{1,r_11}x_1 + B_1 u + D_{1,r_1w}w + D_{1,r_1v}v \\ \delta &= x_1; \quad y = x_1 + E_{1,w}w + E_{1,v}v \end{aligned}$$
where $x = [x_1' \cdots x_{r_1}']'$; $x_i \in \mathbb{R}^{m_1}$, $i = 1, \ldots, r_1$; $x(0) = x_0 := [x_{1,0}' \cdots x_{r_1,0}']' \in \mathcal{D}_{x0}$. Note that, by (6),
$$y = \hat{E}_{1,w}x_1 + E_{1,w}C_2\eta + \hat{E}_{1,v}v =: \bar{x}_1 + \hat{E}_{1,v}v$$
Define $\bar{x}_i = \hat{E}_{1,w}x_i$, $i = 2, \ldots, r_1$. Then, S admits the following representation, in $\xi = [\eta' \, \bar{x}_1' \cdots \bar{x}_{r_1}']' = T_1^{-1}[x' \, \eta']'$ coordinates with $[\eta_0' \, \bar{x}_{1,0}' \cdots \bar{x}_{r_1,0}']' =$

$\xi(0) = \xi_0 \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0})$,

$$\begin{aligned}\dot{\eta} &= A_2\eta + B_2 x_1 + D_2 v = \bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}\bar{x}_1 + D_2 v\\ &= \bar{A}_2\eta + \bar{B}_2\bar{x}_1 + D_2 v \qquad (9a)\\ \dot{\bar{x}}_1 &= E_{1,w}C_2\left(\bar{A}_2\eta + \bar{B}_2\bar{x}_1 + D_2 v\right) + \hat{E}_{1,w}\left(A_{1,11}x_1\right.\\ &\quad \left.+x_2 + D_{1,1w}\left(C_2\eta + K_2 x_1 + E_2 v\right) + D_{1,1v}v\right)\\ &=: \bar{A}_{1,1\eta}\eta + \bar{A}_{1,11}\bar{x}_1 + \bar{x}_2 + \bar{D}_{1,1v}v \qquad (9b)\\ \dot{\bar{x}}_i &= \hat{E}_{1,w}\left(A_{1,i1}x_1 + x_{i+1} + D_{1,iw}\left(C_2\eta + K_2 x_1\right.\right.\\ &\quad \left.\left.+E_2 v\right) + D_{1,iv}v\right); \quad i = 2,\dots,r_1-1\\ &=: \bar{A}_{1,i\eta}\eta + \bar{A}_{1,i1}\bar{x}_1 + \bar{x}_{i+1} + \bar{D}_{1,iv}v; \qquad (9c)\\ \dot{\bar{x}}_{r_1} &= \hat{E}_{1,w}\left(A_{1,r_1 1}x_1 + B_1 u + D_{1,r_1 w}\left(C_2\eta + K_2 x_1\right.\right.\\ &\quad \left.\left.+E_2 v\right) + D_{1,r_1 v}v\right)\\ &=: \bar{A}_{1,r_1\eta}\eta + \bar{A}_{1,r_1 1}\bar{x}_1 + \hat{E}_{1,w}B_1 u + \bar{D}_{1,r_1 v}v \qquad (9d)\\ y &= \bar{x}_1 + \hat{E}_{1,v}v \qquad (9e)\end{aligned}$$

We will further distinguish two exhaustive and mutually exclusive subcases: Case 3a: $n_2 \in \mathbb{N}$; Case 3b: $n_2 = 0$.

Case 3a: $n_2 \in \mathbb{N}$. Clearly, (9) is in form (1) with $m_o = 1$, and admits UVRD r_1 from u to y, since $\hat{E}_{1,w}B_1$ is invertible. We will apply Lemma 1 to prove this subcase. $\forall c_w \geq 0$, $\forall \xi_0 \in \bar{\mathcal{D}}_0$ with $|\xi_0| \leq c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \leq c_w$, $\forall \bar{x}_{1[0,\infty)} \in \mathcal{C}$ with $\|\bar{x}_{1[0,\infty)}\|_\infty \leq c_w$, let $\eta_{[0,\infty)}$ be the solution to (9a). We have $\eta_0 \in \mathcal{D}_{\eta 0}$ and $|\eta_0| \leq c_w$. Let $\tau = \hat{E}_{1,w}^{-1}\bar{x}_1$. Note that $\tau_{[0,\infty)} \in \mathcal{C}$ with $\|\tau_{[0,\infty)}\|_\infty \leq \|\hat{E}_{1,w}^{-1}\|c_w =: c_{w1}$. By the assumption on $\bar{S}_2$, $\exists c_{c1} \geq 0$, which depends only on c_w and c_{w1}, such that $\|\eta_{[0,\infty)}\|_\infty \leq c_{c1}$. Hence, by Lemma 1, (9) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. This proves Case 3a.

Case 3b: $n_2 = 0$. (9) is in the EZDCF (13) of [1]. Then, (9) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ since its EZD is absent. This proves Case 3b and Case 3.

Case 4: $1 = r_1 < \frac{n_1}{m_1}$ and $n_2 \in \mathbb{Z}_+$. By Lemma 2 of [1], without loss of generality, assume that S_1 is given in the following EZDCF representation, B_1 being invertible,

$$\begin{aligned}\dot{x}_z &= A_{1,z}x_z + A_{1,z1}x_1 + D_{1,zw}w + D_{1,zv}v\\ \dot{x}_1 &= A_{1,1z}x_z + A_{1,11}x_1 + B_1 u + D_{1,1w}w + D_{1,1v}v\\ \delta &= x_1; \quad y = x_1 + E_{1,w}w + E_{1,v}v\end{aligned}$$

where $x = [x_z' \; x_1']'$; $x_z \in \mathbb{R}^{n_{z1}}$, $n_{z1} = n_1 - r_1 m_1$; $x_1 \in \mathbb{R}^{m_1}$; and $x(0) = x_0 := \left[x_{z0}' \; x_{1,0}'\right]' \in \mathcal{D}_{x0}$. Note that, by (6), $y = \hat{E}_{1,w}x_1 + E_{1,w}C_2\eta + \hat{E}_{1,v}v =: \bar{x}_1 + \hat{E}_{1,v}v$. The composite system S admits the following representation, in $\xi = [x_z' \; \eta' \; \bar{x}_1']' =: T_1^{-1}[x' \; \eta']'$ coordinates with $\xi(0) = \xi_0 = \left[x_{z0}' \; \eta_0' \; \bar{x}_{1,0}'\right]' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0})$,

$$\begin{aligned}\dot{x}_z &= A_{1,z}x_z + A_{1,z1}\hat{E}_{1,w}^{-1}\left(\bar{x}_1 - E_{1,w}C_2\eta\right) + D_{1,zw}\left(C_2\eta\right.\\ &\quad \left.+K_2 x_1 + E_2 v\right) + D_{1,zv}v\\ &=: \bar{A}_{1,\eta}\eta + A_{1,z}x_z + \bar{A}_{1,z1}\bar{x}_1 + \bar{D}_{1,zv}v \qquad (10a)\\ \dot{\eta} &= A_2\eta + B_2 x_1 + D_2 v = \bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}\bar{x}_1 + D_2 v \qquad (10b)\\ \dot{\bar{x}}_1 &= E_{1,w}C_2(\bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}\bar{x}_1 + D_2 v) + \hat{E}_{1,w}(A_{1,1z}x_z +\\ &\quad A_{1,11}x_1 + B_1 u + D_{1,1w}(C_2\eta + K_2 x_1 + E_2 v) + D_{1,1v}v)\\ &=: \bar{A}_{1,1z}x_z + \bar{A}_{1,1\eta}\eta + \bar{A}_{1,11}\bar{x}_1 + \hat{E}_{1,w}b_1 u + \bar{D}_{1,1v}v \qquad (10c)\\ y &= \bar{x}_1 + \hat{E}_{1,v}v \qquad (10d)\end{aligned}$$

Clearly, (10) is in the EZDCF (11) of [1], and admits UVRD $1 = r_1$ from u to y, since $\hat{E}_{1,w}B_1$ being invertible. Note that (10) is also in the form (1) with $m_o = 1$.

$\forall c_w \geq 0$, $\forall \xi_0 \in \bar{\mathcal{D}}_0$ with $|\xi_0| \leq c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \leq c_w$, $\forall \bar{x}_{1[0,\infty)} \in \mathcal{C}$ with $\|\bar{x}_{1[0,\infty)}\|_\infty \leq c_w$, let $x_{z[0,\infty)}$ and $\eta_{[0,\infty)}$ be the solution to (10a) and (10b). Then, $T_1\xi_0 \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, $(x_{z0}, \hat{E}_{1,w}^{-1}(\bar{x}_{1,0} - E_{1,w}C_2\eta_0)) \in \mathcal{D}_{x0}$ with $|(x_{z0}, \hat{E}_{1,w}^{-1}(\bar{x}_{1,0} - E_{1,w}C_2\eta_0))| \leq c_1 c_w$, where $c_1 := \left\| \begin{bmatrix} I_{n_{z1}} & \mathbf{0}_{n_{z1}\times n_2} & \mathbf{0}_{n_{z1}\times m_1} \\ \mathbf{0}_{m_1\times n_{z1}} & \hat{E}_{1,w}^{-1}E_{1,w}C_2 & \hat{E}_{1,w}^{-1} \end{bmatrix} \right\|$, and $\eta_0 \in \mathcal{D}_{\eta 0}$ with $|\eta_0| \leq c_w$. Let $\tau = \hat{E}_{1,w}^{-1}\bar{x}_1$. Note that $\tau_{[0,\infty)} \in \mathcal{C}$ with $\|\tau_{[0,\infty)}\|_\infty \leq \|\hat{E}_{1,w}^{-1}\|c_w =: c_{w1}$. By the assumption on $\bar{S}_2$, $\exists c_{c1} \geq 0$, which depends only on c_w and c_{w1}, such that $\|\eta_{[0,\infty)}\|_\infty \leq c_{c1}$. Note that $x_{1[0,\infty)} \in \mathcal{C}$ with $\|x_{1[0,\infty)}\|_\infty = \|\hat{E}_{1,w}^{-1}(\bar{x}_{1[0,\infty)} - E_{1,w}C_2\eta_{[0,\infty)})\|_\infty \leq \|\hat{E}_{1,w}^{-1}\|c_w + \|\hat{E}_{1,w}^{-1}E_{1,w}C_2\|c_{c1} =: c_{w2}$; and $w_{[0,\infty)} \in \mathcal{C}$ with $\|w_{[0,\infty)}\|_\infty = \|C_2\eta_{[0,\infty)} + K_2 x_{1[0,\infty)} + E_2 v_{[0,\infty)}\|_\infty \leq \|C_2\|c_{c1} + \|K_2\|c_{w2} + \|E_2\|c_w =: c_{w3}$; and $y_{[0,\infty)} \in \mathcal{C}$ with $\|y_{[0,\infty)}\|_\infty = \|\bar{x}_{1[0,\infty)} + \hat{E}_{1,v}v_{[0,\infty)}\|_\infty \leq c_w + \|\hat{E}_{1,v}\|c_w =: c_{w4}$. Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{C}([0,\infty),\mathbb{R}^{q_w}) \times \mathcal{W}_d$, then, by Definition 1 of [1], $\exists c_{c2} \geq 0$, which depends only on $c_1 c_w$, c_{w4}, c_{w3}, and c_w, such that $\|x_{z[0,\infty)}\|_\infty \leq c_{c2}$. Then, $\|\xi_{z[0,\infty)}\|_\infty \leq \sqrt{c_{c1}^2 + c_{c2}^2}$, where $\xi_z = [x_z' \; \eta']'$. Hence, (10) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ by Lemma 1. This proves Case 4.

Case 5: $2 \leq r_1 < \frac{n_1}{m_1}$ and $n_2 \in \mathbb{Z}_+$. By Lemma 2 of [1], without loss of generality, assume that S_1 is given in the following EZDCF representation, B_1 being invertible,

$$\begin{aligned}\dot{x}_z &= A_{1,z}x_z + A_{1,z1}x_1 + D_{1,zw}w + D_{1,zv}v\\ \dot{x}_i &= A_{1,i1}x_1 + x_{i+1} + D_{1,iw}w + D_{1,iv}v; \;\; 1 \leq i < r_1\\ \dot{x}_{r_1} &= A_{1,r_1 z}x_z + A_{1,r_1 1}x_1 + B_1 u + D_{1,r_1 w}w + D_{1,r_1 v}v\\ \delta &= x_1; \quad y = x_1 + E_{1,w}w + E_{1,v}v\end{aligned}$$

where $x = [x_z' \; x_1' \cdots x_{r_1}']'$; $x_z \in \mathbb{R}^{n_{z1}}$, $n_{z1} = n_1 - m_1 r_1$; $x_i \in \mathbb{R}^{m_1}$, $i = 1,\dots,r_1$; and $x(0) = x_0 = \left[x_{z0}' \; x_{1,0}' \cdots x_{r_1,0}'\right]' \in \mathcal{D}_{x0}$. Note that, by (6),

$$y = \hat{E}_{1,w}x_1 + E_{1,w}C_2\eta + \hat{E}_{1,v}v =: \bar{x}_1 + \hat{E}_{1,v}v$$

Define $\bar{x}_i = \hat{E}_{1,w}x_i$, $i = 2,\dots,r_1$. The composite system S admits the following representation, in $\xi = [x_z' \; \eta' \; \bar{x}_1' \cdots \bar{x}_{r_1}']' = T_1^{-1}[x' \; \eta']'$ coordinates with $\xi(0) = \xi_0 := \left[x_{z0}' \; \eta_0' \; \bar{x}_{1,0}' \cdots \bar{x}_{r_1,0}'\right]' \in \bar{\mathcal{D}}_0 := T_1^{-1}(\mathcal{D}_{x0} \times \mathcal{D}_{\eta 0})$,

$$\begin{aligned}\dot{x}_z &= A_{1,z}x_z + A_{1,z1}\hat{E}_{1,w}^{-1}\left(\bar{x}_1 - E_{1,w}C_2\eta\right) + D_{1,zw}\left(C_2\eta\right.\\ &\quad \left.+K_2 x_1 + E_2 v\right) + D_{1,zv}v\\ &=: A_{1,z}x_z + \bar{A}_{1,\eta}\eta + \bar{A}_{1,z1}\bar{x}_1 + \bar{D}_{1,zv}v \qquad (11a)\\ \dot{\eta} &= A_2\eta + B_2 x_1 + D_2 v = \bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}\bar{x}_1 + D_2 v \qquad (11b)\\ \dot{\bar{x}}_1 &= E_{1,w}C_2\left(\bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}\bar{x}_1 + D_2 v\right) + \hat{E}_{1,w}\left(A_{1,11}x_1\right.\\ &\quad \left.+x_2 + D_{1,1w}\left(C_2\eta + K_2 x_1 + E_2 v\right) + D_{1,1v}v\right)\\ &=: \bar{A}_{1,1\eta}\eta + \bar{A}_{1,11}\bar{x}_1 + \bar{x}_2 + \bar{D}_{1,1v}v \qquad (11c)\\ \dot{\bar{x}}_i &= \hat{E}_{1,w}(A_{1,i1}x_1 + x_{i+1} + D_{1,iv}v + D_{1,iw}(C_2\eta + K_2 x_1 +\\ &\quad E_2 v)) =: \bar{A}_{1,i\eta}\eta + \bar{A}_{1,i1}\bar{x}_1 + \bar{x}_{i+1} + \bar{D}_{1,iv}v; \; 2 \leq i < r_1 \qquad (11d)\\ \dot{\bar{x}}_{r_1} &= \hat{E}_{1,w}\left(A_{1,r_1 z}x_z + A_{1,r_1 1}x_1 + B_1 u + D_{1,r_1 w}\left(C_2\eta\right.\right.\\ &\quad \left.\left.+K_2 x_1 + E_2 v\right) + D_{1,r_1 v}v\right) =: \bar{A}_{1,r_1 z}x_z + \bar{A}_{1,r_1\eta}\eta\\ &\quad +\bar{A}_{1,r_1 1}\bar{x}_1 + \hat{E}_{1,w}B_1 u + \bar{D}_{1,r_1 v}v \qquad (11e)\\ y &= \bar{x}_1 + \hat{E}_{1,v}v \qquad (11f)\end{aligned}$$

Clearly, (11) is in the form (1) with $m_o = 1$, and admits UVRD r_1 from u to y, since $\hat{E}_{1,w}b_1$ is invertible.

We will apply Lemma 1 to prove that (11) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$. $\forall c_w \geq 0$, $\forall \xi_0 \in \bar{\mathcal{D}}_0$ with $|\xi_0| \leq c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \leq c_w$, $\forall \bar{x}_{1[0,\infty)} \in \mathcal{C}$ with $\|\bar{x}_{1[0,\infty)}\|_\infty \leq c_w$, let $x_{z[0,\infty)}$ and $\eta_{[0,\infty)}$ be the solution to (11a) and (11b). Then, $T_1\xi_0 \in \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$, $\eta_0 \in \mathcal{D}_{\eta 0}$ with $|\eta_0| \leq c_w$, and $(x_{z0}, \hat{E}_{1,w}^{-1}(\bar{x}_{1,0} - E_{1,w}C_2\eta_0), \hat{E}_{1,w}^{-1}\bar{x}_{2,0}, \dots, \hat{E}_{1,w}^{-1}\bar{x}_{r_1,0}) =: T_{1x}\xi_0 \in \mathcal{D}_{x0}$

with $|T_{1x}\xi_0| \le \|T_{1x}\| c_w =: c_1 c_w$. Let $\tau = \hat{E}_{1,w}^{-1}\bar{x}_1$. Note that $\tau_{[0,\infty)} \in \mathcal{C}$ with $\|\tau_{[0,\infty)}\|_\infty \le \|\hat{E}_{1,w}^{-1}\| c_w =: c_{w1}$. By the assumption on $\bar{S}_2$, $\exists c_{c1} \ge 0$, which depends only on c_w and c_{w1}, such that $\|\eta_{[0,\infty)}\|_\infty \le c_{c1}$. Note that $x_{1[0,\infty)} \in \mathcal{C}$ with $\|x_{1[0,\infty)}\|_\infty = \|\hat{E}_{1,w}^{-1}(\bar{x}_{1[0,\infty)} - E_{1,w}C_2\eta_{[0,\infty)})\|_\infty \le \|\hat{E}_{1,w}^{-1}\| c_w + \|\hat{E}_{1,w}^{-1}E_{1,w}C_2\| c_{c1} =: c_{w2}$; and $w_{[0,\infty)} \in \mathcal{C}$ with $\|w_{[0,\infty)}\|_\infty = \|C_2\eta_{[0,\infty)} + K_2 x_{1[0,\infty)} + E_2 v_{[0,\infty)}\|_\infty \le \|C_2\| c_{c1} + \|K_2\| \cdot c_{w2} + \|E_2\| c_w =: c_{w3}$; and $y_{[0,\infty)} \in \mathcal{C}$ with $\|y_{[0,\infty)}\|_\infty = \|\bar{x}_{1[0,\infty)} + \hat{E}_{1,v}v_{[0,\infty)}\|_\infty \le c_w + \|\hat{E}_{1,v}\| c_w =: c_{w4}$. Since S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{C}([0,\infty), \mathbb{R}^{q_w}) \times \mathcal{W}_d$, then, by Definition 1 of [1], $\exists c_{c2} \ge 0$, which depends only on $c_1 c_w$, c_{w4}, c_{w3}, and c_w, such that $\|x_{z[0,\infty)}\|_\infty \le c_{c2}$. Then, $\|\xi_{z[0,\infty)}\|_\infty \le \sqrt{c_{c1}^2 + c_{c2}^2}$, where $\xi_z = [x_z' \; \eta']'$. Hence, (11) is minimum phase with respect to $\bar{\mathcal{D}}_0$ and $\mathcal{W}_d$ by Lemma 1. This proves Case 5.

Case 6: $0 = r_1 = \frac{n_1}{m_1}$ and $n_2 \in \mathbb{Z}_+$. By Definition 2 of [1], without loss of generality, assume that the system S_1 is given in the following EZDCF representation, K_1 being invertible,

$$S_1 : y = K_1 u + E_{1,w}w + E_{1,v}v; \quad \delta = K_1 u$$

By (6), we have

$$\begin{aligned} y &= E_{1,w}C_2\eta + \hat{E}_{1,w}K_1 u + \hat{E}_{1,v}v \\ \delta &= K_1 u = \hat{E}_{1,w}^{-1}(y - E_{1,w}C_2\eta - \hat{E}_{1,v}v) \end{aligned}$$

Then, the composite system S admits the EZDCF:

$$\begin{aligned} \dot{\eta} &= A_2\eta + B_2\hat{E}_{1,w}^{-1}(y - E_{1,w}C_2\eta - \hat{E}_{1,v}v) + D_2 v \\ &= \bar{A}_2\eta + B_2\hat{E}_{1,w}^{-1}(y - \hat{E}_{1,v}v) + D_2 v \qquad (12a) \end{aligned}$$

$$y = E_{1,w}C_2\eta + \hat{E}_{1,w}K_1 u + \hat{E}_{1,v}v \qquad (12b)$$

Clearly, (12) admits UVRD $0 = r_1$ from u to y, since $\hat{E}_{1,w}K_1$ is invertible. If $n_2 = 0$, then (12) is minimum phase with respect to $\mathbb{R}^0 = \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$ since its EZD is absent. If $n_2 \in \mathbb{N}$, then (12) is in the form of (14) of [1] with (12a) defining its EZD. $\forall c_w \ge 0$, $\forall \eta_0 \in \mathcal{D}_{\eta 0}$ with $|\eta_0| \le c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \le c_w$, $\forall y_{[0,\infty)} \in \mathcal{C}$ with $\|y_{[0,\infty)}\|_\infty \le c_w$, let $\eta_{[0,\infty)}$ be the solution to (12a). Let $\tau = \hat{E}_{1,w}^{-1}(y - \hat{E}_{1,v}v)$. Note that $\tau_{[0,\infty)} \in \mathcal{C}$ with $\|\tau_{[0,\infty)}\|_\infty \le \|\hat{E}_{1,w}^{-1}\| c_w + \|\hat{E}_{1,w}^{-1}\hat{E}_{1,v}\| c_w =: c_{w1}$. By the assumption on $\bar{S}_2$, $\exists c_c \ge 0$, which depends only on c_w and c_{w1}, such that $\|\eta_{[0,\infty)}\|_\infty \le c_c$. Hence, S is minimum phase with respect to $\bar{\mathcal{D}}_{\eta 0} = \mathcal{D}_{x0} \times \mathcal{D}_{\eta 0}$ and $\mathcal{W}_d$ by Definition 1 of [1]. This proves Case 6.

This completes the proof of the theorem. ■

IV. Bounding lemmas on the inversion of minimum phase systems

In this section, we present results on boundedness of the inverse of minimum phase systems. First, we present a result for a square MIMO LTI system with UVRD *zero*.

Proposition 1: Consider a square MIMO LTI system:

$$\dot{x} = Ax + Bu + D_w w + D_v v; \qquad x(0) = x_0 \quad (13a)$$

$$y = Cx + Ku + E_w w + E_v v \qquad (13b)$$

where $x \in \mathbb{R}^n$ is the state vector, $n \in \mathbb{Z}_+$; $u \in \mathbb{R}^m$ is the control input, $m \in \mathbb{N}$; $y \in \mathbb{R}^m$ is the output; $w \in \mathbb{R}^{q_w}$ is the disturbance input, $q_w \in \mathbb{Z}_+$; $v \in \mathbb{R}^{q_v}$ is the disturbance input, $q_v \in \mathbb{Z}_+$; $x_0 \in \mathcal{D}_0 \subseteq \mathbb{R}^n$, $\mathcal{D}_0$ is a subspace; $v_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_{q_v}$; and all matrices are constant.

Assume (13) admits UVRD 0 from u to y and is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{C}([0,\infty), \mathbb{R}^{q_w}) \times \mathcal{W}_d$, where w and v are viewed as disturbances. Then, $\forall c_w \ge 0$, $\exists c_c \ge 0$, $\forall x_0 \in \mathcal{D}_0$ with $|x_0| \le c_w$; $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \le c_w$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$, $\forall w_{[0,t_f)} \in \mathcal{C}$ with $\|w_{[0,t_f)}\|_\infty \le c_w$; let $x_{[0,t_f)}$ and $y_{[0,t_f)}$ be the solution to (13); if $\|y_{[0,t_f)}\|_\infty \le c_w$, we have $\|x_{[0,t_f)}\|_\infty \le c_c$; and $\|u_{[0,t_f)}\|_\infty \le c_c$.

Proof: We will distinguish 2 exhaustive and mutually exclusive cases: Case 1: $n = 0$; Case 2: $n \in \mathbb{N}$.

Case 1: $n = 0$. By Definition 2 of [1], (13) admits the following EZDCF, K being invertible, $y = Ku + E_w w + E_v v$. Then, we have $u = K^{-1}(y - E_w w - E_v v)$. Clearly, the desired result holds with $c_c := \|K^{-1}\| c_w + \|K^{-1}E_w\| c_w + \|K^{-1}E_v\| c_w$. This proves Case 1.

Case 2: $n \in \mathbb{N}$. By Lemma 4 of [1], without loss of generality, assume that (13) is given in the following EZDCF, B_0 being invertible,

$$\dot{x} = \bar{A}x + \bar{B}(y - E_w w - E_v v) + D_w w + D_v v \quad (14a)$$

$$y = Cx + B_0 u + E_w w + E_v v \qquad (14b)$$

$\forall c_w \ge 0$, $\forall x_0 \in \mathcal{D}_0$ with $|x_0| \le c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \le c_w$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$, $\forall w_{[0,t_f)} \in \mathcal{C}$ with $\|w_{[0,t_f)}\|_\infty \le c_w$; let $x_{[0,t_f)}$ and $y_{[0,t_f)}$ be the solution to (14); and let $\|y_{[0,t_f)}\|_\infty \le c_w$. Since (13) is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{C}([0,\infty), \mathbb{R}^{q_w}) \times \mathcal{W}_d$, by Definition 1 of [1] and Lemma 4, then, $\exists c_{c1} \ge 0$, which depends only on c_w, we have $\|x_{[0,t_f)}\|_\infty \le c_{c1}$. Note that $u = B_0^{-1}(y - Cx - E_w w - E_v v)$. Then, $\|u_{[0,t_f)}\|_\infty \le \|B_0^{-1}\| c_w + \|B_0^{-1}C\| c_{c1} + \|B_0^{-1}E_w\| c_w + \|B_0^{-1}E_v\| c_w =: c_{c2}$. This proves Case 2.

This completes the proof of the proposition. ■

Next, we present results for systems with positive UVRDs. First, we present a technical lemma.

Lemma 2: Consider the following chain of integrators:

$$\dot{x}_i = x_{i+1} + D_{iw}w + D_{iv}v; \; i = 1, \ldots, r_1 - 1 \quad (15a)$$

$$\dot{x}_{r_1} = B_1 u + D_{r_1 w}w + D_{r_1 v}v \qquad (15b)$$

where $x_i \in \mathbb{R}^m$, $i = 1, \ldots, r_1$, $r_1 \in \mathbb{N}$, $m \in \mathbb{N}$; $w \in \mathbb{R}^{q_w}$ is the disturbance input, $q_w \in \mathbb{Z}_+$; $v \in \mathbb{R}^{q_v}$ is the disturbance input, $q_v \in \mathbb{Z}_+$; $u \in \mathbb{R}^m$ is the control input; B_1 is invertible; $x = [x_1' \cdots x_{r_1}']'$; $x(0) = x_0 \in \mathbb{R}^{r_1 m}$; $v_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_{q_v}$; and D_{iw} and D_{iv}, $i = 1, \ldots, r_1$, are constant matrices. Consider another LTI system, S_η, sharing the same set of inputs as (15):

$$\dot{\eta} = A\eta + Bu + D_w w + D_v v; \qquad \eta(0) = \eta_0 \quad (16a)$$

$$y = C\eta \qquad (16b)$$

where $\eta \in \mathbb{R}^{n_2}$ is the state, $n_2 \in \mathbb{N}$; $y \in \mathbb{R}^{m_2}$ is the output; u, w, and v are as (15); $\eta_0 \in \mathcal{D}_0 \subseteq \mathbb{R}^{n_2}$, $\mathcal{D}_0$ is a subspace. Assume all outputs of S_η has relative degree at least r_1 from u and satisfies: $\forall c_w \ge 0$, $\exists c_{c1} \ge 0$, $\forall \eta_0 \in \mathcal{D}_0$ with $|\eta_0| \le c_w$, $\forall u_{[0,\infty)} \in \mathcal{C}$ with $\|u_{[0,\infty)}\|_\infty \le c_w$, $\forall w_{[0,\infty)} \in \mathcal{C}$ with $\|w_{[0,\infty)}\|_\infty \le c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \le c_w$, we have $\|\eta_{[0,\infty)}\|_\infty \le c_{c1}$.

Then, $\forall c_w \ge 0$, $\exists c_{c2} \ge 0$, $\forall x_0 \in \mathbb{R}^{r_1 m}$ with $|x_0| \le c_w$, $\forall \eta_0 \in \mathcal{D}_0$ with $|\eta_0| \le c_w$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$, $\forall w_{[0,t_f)} \in \mathcal{C}$ with $\|w_{[0,t_f)}\|_\infty \le c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \le c_w$. Let $\eta_{[0,t_f)}$ and $y_{[0,t_f)}$ be the solutions to (16), and $x_{[0,t_f)}$ be the solution to (15). If $\|x_{1[0,t_f)}\|_\infty \le c_w$, then $\|y_{[0,t_f)}\|_\infty \le c_{c2}$.

Proof: $\forall c_w \ge 0$, $\forall x_0 \in \mathbb{R}^{r_1 m}$ with $|x_0| \le c_w$, $\forall \eta_0 \in \mathcal{D}_0$ with $|\eta_0| \le c_w$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$, $\forall w_{[0,t_f)} \in \mathcal{C}$ with $\|w_{[0,t_f)}\|_\infty \le c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $c_w \ge \|v_{[0,\infty)}\|_\infty$. Let $\eta_{[0,t_f)}$ and $y_{[0,t_f)}$ be the solutions to (16), and $x_{[0,t_f)}$ be the solution to (15). Let $\|x_{1[0,t_f)}\|_\infty \le$

c_w.

Let $z := \eta - \sum_{k=1}^{r_1} A^{r_1-k}BB_1^{-1}x_k : [0,t_f) \to \mathbb{R}^{n_2}$. Then, z satisfies the following state space equation:

$$\begin{aligned}\dot{z} &= Az - BB_1^{-1}(D_{r_1w}w + D_{r_1v}v) + A^{r_1}BB_1^{-1}x_1 + D_v v \\ &\quad + D_w w - \sum_{k=1}^{r_1-1} A^{r_1-k}BB_1^{-1}(D_{kw}w + D_{kv}v) \\ z(0) &= \eta_0 - \sum_{k=1}^{r_1} A^{r_1-k}BB_1^{-1}x_{k,0}\end{aligned}$$

where $x_0 = \left[x'_{1,0} \cdots x'_{r_1,0}\right]'$; $x_{i,0} \in \mathbb{R}^m$, $i = 1,\dots,r_1$. Then, by the linearity and uniqueness of solution to linear differential equations, $z_{[0,t_f)}$ may be generated by

$$\begin{aligned}&\dot{\xi}_1 = A\xi_1 - BB_1^{-1}(D_{r_1w}w + D_{r_1v}v) + D_w w + D_v v; \\ &\qquad \xi_1(0) = \eta_0 \\ &\dot{\delta}_i = A\delta_i;\ \delta_i(0) = -A^iBB_1^{-1}x_{r_1-i,0};\ i = 0,\dots,r_1-1 \\ &\dot{\zeta}_i = A\zeta_i - A^iBB_1^{-1}(D_{r_1-i\,w}w + D_{r_1-i\,v}v); \\ &\qquad \zeta_i(0) = \mathbf{0}_{n_2\times 1};\ i = 1,\dots,r_1-1 \\ &\dot{\zeta}_{r_1} = A\zeta_{r_1} + A^{r_1}BB_1^{-1}x_1;\ \zeta_{r_1}(0) = \mathbf{0}_{n_2\times 1} \\ &z_{[0,t_f)} = \xi_{1[0,t_f)} + \sum_{i=0}^{r_1-1}\delta_{i[0,t_f)} + \sum_{i=1}^{r_1}\zeta_{i[0,t_f)}\end{aligned}$$

Let $\bar{u}_{[0,t_f)} := -B_1^{-1}(D_{r_1w}w_{[0,t_f)} + D_{r_1v}v_{[0,t_f)})$. Note that $\eta_0 \in \mathcal{D}_0$ and $|\eta_0| \le c_w$; $\bar{u}_{[0,t_f)} \in \mathcal{C}$ with $\|\bar{u}_{[0,t_f)}\|_\infty \le \|B_1^{-1}D_{r_1w}\|c_w + \|B_1^{-1}D_{r_1v}\|c_w =: c_{w1}$; $w_{[0,t_f)} \in \mathcal{C}$ with $\|w_{[0,t_f)}\|_\infty \le c_w$; and $v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \le c_w$. By the assumption on S_η and Lemma 4, $\exists c_{ca} \ge 0$, which depends only on c_w and c_{w1}, such that $\|\xi_{1[0,t_f)}\|_\infty \le c_{ca}$.

By the assumption on S_η and Lemma 9 of [5], we have that the following dynamics $\dot{\kappa}_1 = A\kappa_1 + B\rho$, $\kappa_1(0) = \mathbf{0}_{n_2\times 1}$, is BIBS stable. By repeated application of Lemma 3, we have that the following dynamics

$$\dot{\kappa}_i = A\kappa_i + A^{i-1}B\rho;\ \kappa_i(0) = \mathbf{0}_{n\times 1};\ i = 2,\dots,r_1+1$$

are BIBS stable. Fix any $i = 1,\dots,r_1-1$. Let $\rho_i := -B_1^{-1}\cdot(D_{r_1-i\,w}w + D_{r_1-i\,v}v) : [0,t_f) \to \mathbb{R}^m$. Then, $\rho_{i[0,t_f)} \in \mathcal{C}$ and $\|\rho_{i[0,t_f)}\|_\infty \le \|B_1^{-1}D_{r_1-i\,w}\|c_w + \|B_1^{-1}D_{r_1-i\,v}\|c_w =: c_{wAi}c_w$. Since κ_{i+1} dynamics is BIBS stable, then, ζ_i dynamics is BIBS stable. By Lemma 6 of [5], $\exists c_{Ai} \ge 0$, which depends only on A and A^iB, such that $\|\zeta_{i[0,t_f)}\|_\infty \le c_{Ai}c_{wAi}c_w$. Since κ_{r_1+1} dynamics is BIBS stable, then, ζ_{r_1} dynamics is BIBS stable. Let $\rho_{r_1} := B_1^{-1}x_1$. Then $\rho_{r_1[0,t_f)} \in \mathcal{C}$ and $\|\rho_{r_1[0,t_f)}\|_\infty \le \|B_1^{-1}\|c_w =: c_{wAr_1}c_w$. Then, by Lemma 6 of [5], $\exists c_{Ar_1} \ge 0$, which depends only on A and $A^{r_1}B$, such that $\|\zeta_{r_1[0,t_f)}\|_\infty \le c_{Ar_1}c_{wAr_1}c_w$.

$\forall i = 0,\dots,r_1-1$. Since κ_{i+1} system is BIBS stable, by Lemma 3, note that $\left|-B_1^{-1}x_{r_1-i,0}\right| \le \|B_1^{-1}\|c_w = c_{wAr_1}c_w$, then, $\exists c_{Bi} \ge 0$, which depends only on A and A^iB, such that $\|\delta_{i[0,t_f)}\|_\infty \le c_{Bi}c_{wAr_1}c_w$.

Hence, we have $\|z_{[0,t_f)}\|_\infty \le c_{ca} + \sum_{i=0}^{r_1-1} c_{Bi}c_{wAr_1}c_w + \sum_{i=1}^{r_1} c_{Ai}c_{wAi}c_w =: c_{c1}$.

Note that $y = Cz + \sum_{k=1}^{r_1} CA^{r_1-k}BB_1^{-1}x_k = Cz + CA^{r_1-1}BB_1^{-1}x_1$ since y has relative degree at least r_1 from u. Hence, $\|y_{[0,t_f)}\|_\infty \le \|C\|c_{c1} + \|CA^{r_1-1}BB_1^{-1}\|c_w =: c_{c2}$.

This completes the proof of the lemma. ■

Proposition 2: Consider two LTI systems sharing the same inputs:

$$S_1 : \begin{cases}\dot{x} = A_1x + B_1u + D_{1,w}w + D_{1,v}v;\ x(0) = x_0 \\ y_1 = C_1x + E_{1,w}w + E_{1,v}v\end{cases} \tag{17}$$

$$S_2 : \begin{cases}\dot{\eta} = A_2\eta + B_2u + D_{2,w}w + D_{2,v}v;\ \eta(0) = \eta_0 \\ y_2 = C_2\eta\end{cases} \tag{18}$$

where $x \in \mathbb{R}^{n_1}$ is the state of S_1, $n_1 \in \mathbb{N}$; $u \in \mathbb{R}^m$ is the control input, $m \in \mathbb{N}$; $y_1 \in \mathbb{R}^m$ is the output of S_1; $w \in \mathbb{R}^{q_w}$ is the disturbance input, $q_w \in \mathbb{Z}_+$; $v \in \mathbb{R}^{q_v}$ is the disturbance input, $q_v \in \mathbb{Z}_+$; $\eta \in \mathbb{R}^{n_2}$ is the state of S_2, $n_2 \in \mathbb{N}$; $y_2 \in \mathbb{R}^{m_2}$ is the output of S_2; $x_0 \in \mathcal{D}_{x0} \subseteq \mathbb{R}^{n_1}$, $\mathcal{D}_{x0}$ is a subspace; $\eta_0 \in \mathcal{D}_{\eta 0} \subseteq \mathbb{R}^{n_2}$, $\mathcal{D}_{\eta 0}$ is a subspace; $v_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_{q_v}$; and all matrices are constant.

Assume that

(i) S_1 admits UVRD r_1 from u to y_1, $1 \le r_1 \le \frac{n_1}{m}$, and is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{C}([0,\infty),\mathbb{R}^{q_w}) \times \mathcal{W}_d$, where w and v are viewed as disturbance inputs;

(ii) S_2 satisfies that $\forall c_w \ge 0$, $\exists c_{c1} \ge 0$, $\forall \eta_0 \in \mathcal{D}_{\eta 0}$ with $|\eta_0| \le c_w$, $\forall u_{[0,\infty)} \in \mathcal{C}$ with $\|u_{[0,\infty)}\|_\infty \le c_w$, $\forall w_{[0,\infty)} \in \mathcal{C}$ with $\|w_{[0,\infty)}\|_\infty \le c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \le c_w$, we have $\|\eta_{[0,\infty)}\|_\infty \le c_{c1}$; and

(iii) y_2 has relative degree at least $r_2 \in \mathbb{N}$ from u, i.e., $C_2B_2 = \cdots = C_2A_2^{r_2-2}B_2 = \mathbf{0}_{m_2\times m}$.

By Lemmas 2 and 3 of [1], without loss of generality, assume that S_1 is given in the EZDCF: Case 1: $mr_1 = n_1$

$$\dot{x}_i = A_{1,i1}x_1 + x_{i+1} + D_{1,iw}w + D_{1,iv}v;\ 1 \le i < r_1 \tag{19a}$$
$$\dot{x}_{r_1} = A_{1,r_11}x_1 + B_1u + D_{1,r_1w}w + D_{1,r_1v}v \tag{19b}$$
$$y_1 = x_1 + E_{1,w}w + E_{1,v}v \tag{19c}$$

where $x = [x'_1 \cdots x'_{r_1}]'$; $x_i \in \mathbb{R}^m$, $i = 1,\dots,r_1$; $x(0) = x_0 = \left[x'_{1,0} \cdots x'_{r_1,0}\right]' \in \mathcal{D}_{x0}$; and B_1 is invertible; Case 2: $mr_1 < n_1$

$$\dot{x}_z = A_{1,z}x_z + A_{1,z1}x_1 + D_{1,zw}w + D_{1,zv}v \tag{20a}$$
$$\dot{x}_i = A_{1,i1}x_1 + x_{i+1} + D_{1,iw}w + D_{1,iv}v;\ 1 \le i < r_1 \tag{20b}$$
$$\begin{aligned}\dot{x}_{r_1} &= A_{1,r_1z}x_z + A_{1,r_11}x_1 + B_1u + D_{1,r_1w}w \\ &\quad + D_{1,r_1v}v\end{aligned} \tag{20c}$$
$$y_1 = x_1 + E_{1,w}w + E_{1,v}v \tag{20d}$$

where $x = [x'_z\, x'_1 \cdots x'_{r_1}]'$; $x_z \in \mathbb{R}^{n_z}$, $n_z = n_1 - r_1m$; $x_i \in \mathbb{R}^m$, $i = 1,\dots,r_1$; $x(0) = x_0 = \left[x'_{z0}\, x'_{1,0} \cdots x'_{r_1,0}\right]' \in \mathcal{D}_{x0}$; and B_1 is invertible.

Then, $\forall c_w \ge 0$, $\exists c_c \ge 0$, $\forall x_0 \in \mathcal{D}_{x0}$ with $|x_0| \le c_w$, $\forall \eta_0 \in \mathcal{D}_{\eta 0}$ with $|\eta_0| \le c_w$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$, $\forall w_{[0,t_f)} \in \mathcal{C}$ with $\|w_{[0,t_f)}\|_\infty \le c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \le c_w$. Let $\eta_{[0,t_f)}$ and $y_{2[0,t_f)}$ be the solutions to (18), and $x_{[0,t_f)}$ and $y_{1[0,t_f)}$ be the solution to (17). If $\|x_{1[0,t_f)}\|_\infty \le c_w$, …, $\|x_{k_0[0,t_f)}\|_\infty \le c_w$, for some fixed $k_0 \in \{1,\dots,r_1\}$, and $r_2 \ge r_1 - k_0 + 1$, we have $\|y_{2[0,t_f)}\|_\infty \le c_c$.

Proof: $\forall c_w \ge 0$, $\forall x_0 \in \mathcal{D}_{x0}$ with $|x_0| \le c_w$, $\forall \eta_0 \in \mathcal{D}_{\eta 0}$ with $|\eta_0| \le c_w$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$, $\forall w_{[0,t_f)} \in \mathcal{C}$ with $\|w_{[0,t_f)}\|_\infty \le c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \le c_w$. Let $\eta_{[0,t_f)}$ and $y_{2[0,t_f)}$ be the solutions to (18), and $x_{[0,t_f)}$ and $y_{1[0,t_f)}$ be the solution to (17). Let $\|x_{1[0,t_f)}\|_\infty \le c_w$, …, $\|x_{k_0[0,t_f)}\|_\infty \le c_w$.

We will distinguish two exhaustive and mutually exclusive cases: Case 1: $mr_1 = n_1$; Case 2: $mr_1 < n_1$.

Case 1: $mr_1 = n_1$. S_1 admits the representation (19). We will apply Lemma 2 to prove this case. The chain of integrators is

$$\begin{aligned}&\dot{x}_i = x_{i+1} + [A_{1,i1}\, D_{1,iw}]\,[x'_1\, w']' + D_{1,iv}v;\ k_0 \le i < r_1 \\ &\dot{x}_{r_1} = B_1u + [A_{1,r_11}\, D_{1,r_1w}]\,[x'_1\, w']' + D_{1,r_1v}v\end{aligned}$$

Then, by Lemma 2, $\exists c_c \ge 0$, which depends only on c_w, such that $\|y_{2[0,t_f)}\|_\infty \le c_c$. This proves Case 1.

Case 2: $mr_1 < n_1$. S_1 admits the representation (20).

Claim 1: $\exists \bar{c}_c \ge 0$, which depends only on c_w, such that $\|x_{z[0,t_f)}\|_\infty \le \bar{c}_c$.

Proof: By the fact that S_1 is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{C}([0,\infty),\mathbb{R}^{q_w}) \times \mathcal{W}_d$ and Definition 1 of [1] and Lemma 4, $\exists \bar{c}_c \geq 0$, which depends only on c_w, such that $\|x_{z[0,t_f)}\|_\infty \leq \bar{c}_c$. This proves the claim. ■

Now, we will apply Lemma 2 to prove this case. The chain of integrators is

$$\dot{x}_i = x_{i+1} + [\,\mathbf{0}_{m\times n_z}\; A_{1,i1}\; D_{1,iw}\,]\,[x_z'\; x_1'\; w']' + D_{1,iv}v; \quad i = k_0,\ldots,r_1-1$$

$$\dot{x}_{r_1} = B_1 u + [\,A_{1,r_1z}\; A_{1,r_11}\; D_{1,r_1w}\,]\,[x_z'\; x_1'\; w']' + D_{1,r_1v}v$$

Then, by Lemma 2, $\exists c_c \geq 0$, which depends only on c_w and $\bar{c}_c$, such that $\|y_{2[0,t_f)}\|_\infty \leq c_c$. This proves Case 2 and completes the proof of the proposition. ■

In application of Proposition 2, we will refer to S_1 as the reference system. Finally, we present a corollary without proof, which has a stronger assumption on S_2.

Corollary 1: Consider two SISO LTI systems sharing the same inputs (17) and (18) as in Proposition 2.

Assume that

(i) S_1 admits UVRD r_1 from u to y_1, $1 \leq r_1 \leq \frac{n_1}{m}$, and is minimum phase with respect to $\mathcal{D}_{x0}$ and $\mathcal{C}([0,\infty),\mathbb{R}^{q_w}) \times \mathcal{W}_d$, where w and v are viewed as disturbance inputs; and

(ii) the matrix A_2 is Hurwitz and y_2 has relative degree at least $r_2 \in \mathbb{N}$ from u, i.e., $C_2B_2 = \cdots = C_2A_2^{r_2-2}B_2 = \mathbf{0}_{m_2\times m}$.

Let S_1 be given in the EZDCF (19) or (20) depending on whether $mr_1 = n_1$ or $mr_1 < n_1$, respectively.

Then, $\forall c_w \geq 0$, $\exists c_c \geq 0$, $\forall x_0 \in \mathcal{D}_{x0}$ with $|x_0| \leq c_w$, $\forall \eta_0 \in \mathcal{D}_{\eta 0}$ with $|\eta_0| \leq c_w$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall u_{[0,t_f)} \in \mathcal{C}$, $\forall w_{[0,t_f)} \in \mathcal{C}$ with $\|w_{[0,t_f)}\|_\infty \leq c_w$, $\forall v_{[0,\infty)} \in \mathcal{W}_d$ with $\|v_{[0,\infty)}\|_\infty \leq c_w$. Let $\eta_{[0,t_f)}$ and $y_{2[0,t_f)}$ be the solutions to (18), and $x_{[0,t_f)}$ and $y_{1[0,t_f)}$ be the solution to (17). If $\|x_{1[0,t_f)}\|_\infty \leq c_w$, $\ldots$, $\|x_{k_0[0,t_f)}\|_\infty \leq c_w$, for some fixed $k_0 \in \{1,\ldots,r_1\}$, and $r_2 \geq r_1 - k_0 + 1$, we have $\|y_{2[0,t_f)}\|_\infty \leq c_c$.

V. Conclusions

In this paper, we have further investigated properties of minimum phase systems using the generalized definition introduced in [1]. We proved that the composite system consisting of a minimum phase system in feedback connection with another linear system satisfying a certain boundedness condition is itself a minimum phase system. We have established two results (Propositions 1 and 2) on the stable invertibility of minimum phase systems. When a square minimum phase linear system has UVRD *zero* from the control input to the output, has a bounded admissible initial condition, and a bounded admissible disturbance waveform, then the control input and the state trajectory are bounded if the output of the system is bounded. When a square minimum phase linear system has UVRD $r_1 \in \mathbb{N}$, a bounded admissible initial condition, and a bounded admissible disturbance waveform, if the noiseless output together with its noiseless derivatives up to k_0th order are bounded, then, the output of a "stable" linear system sharing the same inputs as the minimum phase system is bounded if the relative degrees of the outputs of the "stable" system satisfy $r_2 \geq r_1 - k_0$.

Future research along this direction lies in the model reference robust adaptive control using the new definition of minimum phase.

Appendix

Lemma 3: Consider the LTI system: $\dot{z} = Az + Bv$, $z(0) = \mathbf{0}_{n\times 1}$, where z is the n-dimensional state, $n \in \mathbb{Z}_+$; and v is the p-dimensional input, $p \in \mathbb{Z}_+$. Assume that the system is BIBS stable. Then, the following statements hold.

1) For system $\dot{\eta} = A\eta$, $\eta(0) = B\xi$, where $\xi \in \mathbb{R}^p$, there exist $k \geq 0$ and $\lambda > 0$ such that $\forall \xi \in \mathbb{R}^p$ with $|\xi| \leq c_w \geq 0$, we have $|\eta(t)| \leq c_w k \mathrm{e}^{-\lambda t}$, $\forall t \in [0,\infty)$.
2) The system $\dot{x} = Ax + ABu$, $x(0) = \mathbf{0}_{n\times 1}$, is BIBS stable.

Proof: By Lemma 6 of [5], $\exists k \geq 0$ and $\exists \lambda > 0$ such that $\|\mathrm{e}^{At}B\| \leq k\mathrm{e}^{-\lambda t}$, $\forall t \geq 0$. 1) By Theorem 4-4 of [7], $|\eta(t)| = \left|\mathrm{e}^{At}B\xi\right| \leq k\mathrm{e}^{-\lambda t}c_w$, $\forall t \geq 0$. 2) we note that $(\sum_{i=0}^{\infty} \frac{1}{i!}(At)^i)AB = \mathrm{e}^{At}AB = A(\sum_{i=0}^{\infty} \frac{1}{i!}(At)^i)B = A\mathrm{e}^{At}B$, $\forall t \in \mathbb{R}$. Then, $\|\mathrm{e}^{At}AB\| = \|A\mathrm{e}^{At}B\| \leq \|A\|\|\mathrm{e}^{At}B\| \leq \|A\|k\mathrm{e}^{-\lambda t}$, $\forall t \geq 0$. By Lemma 6 of [5], the result holds. ■

Lemma 4: Consider the LTI system:

$$\dot{x} = Ax + Bu + Dw; \qquad x(0) = x_0 \tag{21}$$

where x is the n-dimensional state, $n \in \mathbb{Z}_+$; u is the p-dimensional input, $p \in \mathbb{Z}_+$; w is the q-dimensional disturbance input, $q \in \mathbb{Z}_+$; $x_0 \in \mathcal{D}_0 \subseteq \mathbb{R}^n$, $\mathcal{D}_0$ is a subspace; $w_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_q$. Assume that (21) satisfies: $\forall c_w \geq 0$, $\exists c_c \geq 0$, $\forall x_0 \in \mathcal{D}_0$ with $|x_0| \leq c_w$, $\forall u_{[0,\infty)} \in \mathcal{C}$ with $\|u_{[0,\infty)}\|_\infty \leq c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \leq c_w$, we have $\|x_{[0,\infty)}\|_\infty \leq c_c$.

Then, $\forall c_w \geq 0$, if c_c is the constant defined previously, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall x_0 \in \mathcal{D}_0$ with $|x_0| \leq c_w$, $\forall u_{[0,t_f)} \in \mathcal{C}$ with $\|u_{[0,t_f)}\|_\infty \leq c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \leq c_w$, we have $\|x_{[0,t_f)}\|_\infty \leq c_c$.

Proof: $\forall c_w \geq 0$, $\forall t_f \in (0,\infty] \subset \mathbb{R}_e$, $\forall x_0 \in \mathcal{D}_0$ with $|x_0| \leq c_w$, $\forall u_{[0,t_f)} \in \mathcal{C}$ with $\|u_{[0,t_f)}\|_\infty \leq c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \leq c_w$. Let $x_{[0,t_f)}$ be the solution to (21). $\forall t \in [0,t_f)$, we will show that $|x(t)| \leq c_c$. Let $\bar{u}_{[0,\infty)} \in \mathcal{C}$ be given by $\bar{u}(s) = \begin{cases} u(s) & s \in [0,t] \\ u(t) & s \in (t,\infty) \end{cases}$. Clearly, we have $\|\bar{u}_{[0,\infty)}\|_\infty \leq c_w$. Let $\bar{x}_{[0,\infty)}$ be the solution to (21) due to $\bar{u}_{[0,\infty)}$, $w_{[0,\infty)}$, and x_0. Then, $\|\bar{x}_{[0,\infty)}\|_\infty \leq c_c$. By the causality of (21), we have $x_{[0,t]} = \bar{x}_{[0,t]}$. Hence, $|x(t)| \leq c_c$. This completes the proof. ■

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