Generalized Minimum Phase Property for Right-Invertible MIMO LTI System in Tandem with An Integration Block on the Output of the System†

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***Abstract*— In [1], we had introduced a generalized concept of minimum phase for finite-dimensional continuous-time multiple-input and multiple-output (MIMO) linear time-invariant (LTI) systems with additive disturbances. In this paper, we study the minimum phase property of the composite system that comprises of a right invertible MIMO LTI system S in tandem with an integration block on the output of the system. We prove that in this special case, the extended zero dynamics of the composite system is of the same order as the one for S, and the composite system remains minimum phase when S is so to begin with. We also show two other interesting results concerning composite MIMO LTI systems.**

***Index Terms*— continuous-time systems, dynamic extension, minimum phase, extended zero dynamics.**

I. Introduction

The minimum phase property is of paramount importance in model reference control theory, having attracted sustained research attention [2], [3], [4], [5], [6]. In [5], we introduced a generalization of the minimum phase concept for SISO LTI systems with additive distubance inputs which is proved to be necessary for the solvability of the output feedback model reference control problem. A further step in this direction [6] yields the generalized minimum phase concept for finite-dimensional MIMO LTI system with additive disturbance. In [1], the definition of the generalized minimum phase is updated slightly so that we can have stronger statements in the robust model reference control for a class of uncertain systems. It is proved in [7] that this generalized minimum phase concept is necessary for the solvability of the output feedback model reference control problem for MIMO LTI systems. The relationship of the generalized concept of minimum phase with that introduced in [3] and [2] has been investigated. It is shown in [1] that when the MIMO LTI system is minimum phase as defined in [3], then it also satisfies the generalized minimum phase condition of [1]. The converse holds if the MIMO LTI system is stabilizable with respect the the control input. It is also shown in [1] that the generalized minimum phase property, in particular the extended zero dynamics, remains unchanged under a step of the dynamic extension algorithm ([3]).

In this paper, we further investigate the properties of minimum phase MIMO LTI systems with additive disturbance input using the definition in [1]. We first show that, for a MIMO LTI system S, attaching a dynamic/static controller without feedback will not decrease the dimension of the extended zero dynamics. Then, we prove that the composite system that comprises of a right invertible MIMO LTI system S in tandem with an integration block on the output of the system admits an extended zero dynamics that is of the same order as the one for S, and the composite system remains minimum phase when S is so to begin with. Finally, we show that the composite system consisting of a MIMO LTI system S with vector relative degree $(r_1, \ldots, r_m)$, where $m \in \mathbb{Z}_+$ is the dimension of the output, in tandem with a diagonal integration block on the output of S admits vector relative degree $(r_1 + 1, \ldots, r_m + 1)$.

The balance of the paper is as follows. In the next section, we list the notations used in the paper. Then, in Section III, we present the main results of the paper. The paper ends with some concluding remarks in Section IV.

II. Notations

Let $\mathbb{R}$ denote the real line; $\overline{\mathbb{R}_+} := [0, \infty) \subset \mathbb{R}$; $\mathbb{N}$ be the set of natural numbers; $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Unless specified, all signals, constants, and matrices are real. For a continuous function f, we say that it belongs to $\mathcal{C}$. For any $z \in \mathbb{R}^n$, $|z|$ denotes $\sqrt{z'z}$. I_n denotes the $n \times n$-dimensional identity matrix. For any matrix A, $A^0 = I$. For any matrix M, $\|M\|$ denotes its 2-induced norm. $\mathbf{0}_{m\times n}$ denotes the $m \times n$-dimensional matrix whose elements are all zeros. For any waveform $u_{[0,t_f)} \in \mathcal{C}([0, t_f), \mathbb{R}^p)$, where $t_f \geq 0$ and $p \in \mathbb{Z}_+$, $\|u_{[0,t_f)}\|_\infty = \sup_{t\in[0,t_f)} |u(t)|$.

III. Main Result

Consider a finite-dimensional MIMO LTI system S:

$$\dot{x} = Ax + Bu + Dw; \quad x(0) = x_0 \in \mathcal{D}_0 \tag{1a}$$

$$y = Cx + Fu + Ew \tag{1b}$$

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where $x \in \mathbb{R}^n$ is the state vector, $n \in \mathbb{Z}_+$; x_0 is the initial state vector that belongs to the subspace $\mathcal{D}_0 \subseteq \mathbb{R}^n$, $u \in \mathbb{R}^p$ is the control input, $p \in \mathbb{Z}_+$; $y \in \mathbb{R}^m$ is the output, $m \in \mathbb{Z}_+$; and $w \in \mathbb{R}^q$ is the disturbance input, $q \in \mathbb{Z}_+$; $w_{[0,\infty)} \in \mathcal{W}_d$, which is of class $\mathcal{B}_q$; A, B, D, C, F, and E are constant matrices of appropriate dimensions.

By [6], the system S admits the extended zero dynamics that is given by, $x_z = Kx$

$$\dot{x}_z = A_z x_z + A_{z1} y + (KD - A_{z1}E)w; \qquad (2)$$
$$x_z(0) = Kx_0 \in K(\mathcal{D}_0)$$

where $K \in \mathbb{R}^{s\times n}$, $A_z \in \mathbb{R}^{s\times s}$, $A_{z1} \in \mathbb{R}^{s\times m}$ be the maximal solution (in terms of $s \in \mathbb{Z}_+$) to

$$KA = A_z K + A_{z1} C \qquad (3a)$$
$$A_{z1} F = KB \qquad (3b)$$

with the pair (A_z, A_{z1}) being unique for each K. The system S is said to be *minimum phase with respect to* $\mathcal{D}_0$ *and* $\mathcal{W}_d$ if $\forall c_w \in \overline{\mathbb{R}_+}$, $\exists c_c \in \overline{\mathbb{R}_+}$, $\forall x_0 \in \mathcal{D}_0$ with $|x_0| \leq c_w$, $\forall y_{[0,\infty)} \in \mathcal{C}$ with $\|y_{[0,\infty)}\|_\infty \leq c_w$, $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \leq c_w$, we have $\|x_{z[0,\infty)}\|_\infty \leq c_c$.

We first state a result that adding a linear dynamic/static controller without feedback

$$\dot{\zeta} = A_{22}\zeta + B_2 v + D_2 w \qquad (4a)$$
$$u = C_2\zeta + F_2 v + E_2 w \qquad (4b)$$

to (1) will result in a expanded system with an extended zero dynamics of order $\hat{s} \geq s$.

Proposition 1: Consider the MIMO LTI system (1) with a linear dynamic/static controller (4) in tandem, where $\zeta \in \mathbb{R}^{n_1}$ is the state vector of the controller, $n_1 \in \mathbb{Z}_+$; $v \in \mathbb{R}^{p_1}$ is the new control input, $p_1 \in \mathbb{Z}_+$; $u \in \mathbb{R}^p$ is the control input to (1); w is the same as in (1); the matrices A_{22}, B_2, D_2, C_2, F_2, and E_2 are constant and of appropriate dimensions. Then, the composite system consisting of (1) and (4) admits an extended zero dynamics of order $\hat{s} \geq s$, where s is the dimension of extended zero dynamics (2) for (1).

Proof: The composite system has the following state space representation, in $\hat{\lambda} := (x, \zeta)$ coordinates:

$$\dot{\hat{\lambda}} = \hat{A}\hat{\lambda} + \hat{B}v + \hat{D}w \qquad (5a)$$
$$y = \hat{C}\hat{\lambda} + \hat{F}v + \hat{E}w \qquad (5b)$$

where $\hat{A} = \begin{bmatrix} A & BC_2 \\ \mathbf{0}_{n_1\times n} & A_{22} \end{bmatrix}$; $\hat{B} = \begin{bmatrix} BF_2 \\ B_2 \end{bmatrix}$; $\hat{D} = \begin{bmatrix} D + BE_2 \\ D_2 \end{bmatrix}$; $\hat{C} = \begin{bmatrix} C & FC_2 \end{bmatrix}$; $\hat{F} = FF_2$; and $\hat{E} = E + FE_2$. It is easy to show that $\hat{K} := \begin{bmatrix} K & \mathbf{0}_{s\times n_1} \end{bmatrix}$, $\hat{A}_z := A_z$, and $\hat{A}_{z1} := A_{z1}$ satisfies (7) of [1] for the composite system (5). Since K is full row rank, then $\hat{K}$ is of full row rank s. This shows that the maximal solution to (7) of [1] for the composite system (5) must have dimension $\hat{s} \geq s$. This completes the proof of the proposition. ∎

Next consider the system S in tandem with an integration block S_1, define by:

$$\dot{\xi} = A_1\xi + y + D_1 w; \quad \xi(0) =: \xi_0 \in \mathbb{R}^m \qquad (6a)$$
$$\bar{y} = \xi + E_1 w \qquad (6b)$$

where $A_1 \in \mathbb{R}^{m\times m}$, $D_1 \in \mathbb{R}^{m\times q}$ and $E_1 \in \mathbb{R}^{m\times q}$ are constant matrices. For the system S_1, it is clear that the output $\bar{y}$ has uniform vector relative degree of 1 with respect to input y and is minimum phase with respect to $\mathbb{R}^m$ and $\mathcal{W}_d$ since the extended zero dynamics is absent.

We next present a result which shows that the composite system consisting of S in tandem with S_1 is minimum phase with respect to $\mathbb{R}^m \times \mathcal{D}_0$ and $\mathcal{W}_d$, if S is right invertible and minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$.

Theorem 1: Consider composite system with S in tandem with S_1. Assume that the system S is right-invertible. Then, the composite system admits the extended zero dynamics:

$$\begin{aligned}\dot{\lambda}_z = {} & A_z\lambda_z + (A_zA_{z1} - A_{z1}A_1)\bar{y} + (-A_{z1}(D_1 \\ & - A_1E_1))w - A_zA_{z1}E_1w + (KD - A_{z1}E)w; \quad (7) \\ & \lambda_z(0) = -A_{z1}\xi_0 + Kx_0\end{aligned}$$

where $\lambda_z \in \mathbb{R}^s$ is the state vector.

If, in addition, S is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$, then the composite system is minimum phase with respect to $\mathbb{R}^m \times \mathcal{D}_0$ and $\mathcal{W}_d$.

Proof: The composite system admits the state space representation, in $\lambda := (\xi, x) \in \mathbb{R}^{m+n}$ coordinates,

$$\begin{aligned}\dot{\lambda} = {} & \begin{bmatrix} A_1 & C \\ \mathbf{0}_{n\times m} & A \end{bmatrix}\lambda + \begin{bmatrix} F \\ B \end{bmatrix}u + \begin{bmatrix} D_1 + E \\ D \end{bmatrix}w \qquad (8a) \\ =: {} & \bar{A}\lambda + \bar{B}u + \bar{D}w; \quad \lambda(0) = (\xi_0, x_0) \in \mathbb{R}^m \times \mathcal{D}_0\end{aligned}$$

$$\bar{y} = \begin{bmatrix} I_m & \mathbf{0}_{m\times n} \end{bmatrix}\lambda + E_1 w =: \bar{C}\lambda + \bar{F}u + \bar{E}w \qquad (8b)$$

We will first show that the matrices $\bar{K} := \begin{bmatrix} -A_{z1} & K \end{bmatrix} \in \mathbb{R}^{s\times(m+n)}$, $\bar{A}_z := A_z \in \mathbb{R}^{s\times s}$, and $\bar{A}_{z1} := A_zA_{z1} - A_{z1}A_1 \in \mathbb{R}^{s\times m}$ satisfies Equations (7) of [6] for the composite system (8). This is easy to check as following.

$$\begin{aligned}\bar{K}\bar{A} &= \begin{bmatrix} -A_{z1} & K \end{bmatrix}\begin{bmatrix} A_1 & C \\ \mathbf{0}_{n\times m} & A \end{bmatrix} \\ &= \begin{bmatrix} -A_{z1}A_1 & -A_{z1}C + KA \end{bmatrix} = \begin{bmatrix} -A_{z1}A_1 & A_zK \end{bmatrix} \\ &= \bar{A}_z\bar{K} + \bar{A}_{z1}\bar{C}\end{aligned}$$

$$\bar{A}_{z1}\bar{F} = \mathbf{0}_{s\times p} = -A_{z1}F + KB = \bar{K}\bar{B}$$

Clearly, $\mathrm{rank}(\bar{K}) = s$ since $\mathrm{rank}(K) = s$. We need to show this set of solution ($\bar{K}$, $\bar{A}_z$, and $\bar{A}_{z1}$) to (7) of [1] is maximal. By [8] and the right invertibility assumption, S can be dynamic extended to achieve vector relative degree. By [1], the extended zero dynamics is invariant under one step of dynamic extension. Then, S can be dynamic extended to $\acute{S}$ with a dynamic controller of the form (4) such that $\acute{S}$ has uniform vector relative degree $r \in \mathbb{Z}_+$ without changing its extended zero dynamics (2). By Lemma 1 of [1], the dimension s of (2) equals to $\acute{n} - rm$, where $\acute{n}$ is the dimension of the system $\acute{S}$. Note that the composite system of $\acute{S}$ in tandem with S_1 has uniform vector relative degree $r+1$ and dimension $\acute{n}+m$ by [9]. By Lemma 1 of [1], we can deduce that this composite system admits a extended zero dynamics of dimension $\acute{n}+m-(r+1)m = \acute{n}-rm = s$. It is easy to see that the composite system (8) of S in tandem with S_1 together with the dynamic controller (4) forms the composite system of $\acute{S}$ in tandem with S_1. By Proposition 1, the extended zero dynamics for the composite system (8) has dimension less than or equal to the dimension of the extended zero dynamics for the composite system of $\acute{S}$ in tandem with S_1. Thus, the composite system of S in tandem with S_1 admits an extended zero dynamics of dimension less than or equal to s. This shows that the set of solution $\bar{K}$, $\bar{A}_z$, and $\bar{A}_{z1}$ to equations (7) of [1] is maximal.

Define $\lambda_z := \bar{K}\lambda = Kx - A_{z1}\xi$, and λ_z can be shown to admit the dynamics (7). This completes the proof of the first statement.

Consider the second statement in the theorem and assume that S is minimum phase with respect to $\mathcal{D}_0$ and $\mathcal{W}_d$. By Lemma 5 of [6], the system $\dot{z} = A_z z + A_{z1} v$; $z(0) = \mathbf{0}_s$ is BIBS stable. Then, by Lemma 3 of [10], we have the system $\dot{\bar{z}} = A_z \bar{z} + A_z A_{z1} v$; $\bar{z} = \mathbf{0}_s$ is BIBS stable; and the system $\dot{\hat{z}} = A_z \hat{z}$; $\hat{z}(0) = A_{z1}\beta$ has bounded state trajectory, $\forall \beta \in \mathbb{R}^m$. $\forall c_w \in \overline{\mathbb{R}_+}$, $\exists c_c \in \overline{\mathbb{R}_+}$, $\forall \lambda_0 \in \mathbb{R}^m \times \mathcal{D}_0$ with $|\lambda_0| \leq c_w$, $\forall \bar{y}_{[0,\infty)} \in \mathcal{C}$ with $\|\bar{y}_{[0,\infty)}\|_\infty \leq c_w$, , $\forall w_{[0,\infty)} \in \mathcal{W}_d$ with $\|w_{[0,\infty)}\|_\infty \leq c_w$, by linearity of (7), we have

$$\begin{aligned}
\dot{\eta}_1 &= A_z \eta_1 + A_z A_{z1}(\bar{y} - E_1 w);\ \eta_1(0) = \mathbf{0}_s \\
\dot{\eta}_2 &= A_z \eta_2 + A_{z1}(-A_1 \bar{y} + (A_1 E_1 - D_1) w) \\
&\quad +(KD - A_{z1}E)w;\ \eta_2(0) = Kx_0 \\
\dot{\eta}_3 &= A_z \eta_3;\ \eta_3(0) = -A_{z1}\xi_0 \\
&\lambda_{z[0,\infty)} = \eta_{1[0,\infty)} + \eta_{2[0,\infty)} + \eta_{3[0,\infty)}
\end{aligned}$$

Then, $\|\bar{y}_{[0,\infty)} - E_1 w_{[0,\infty)}\|_\infty \leq c_w + \|E_1\| c_w =: c_{w1}$; $\| - A_1 \bar{y}_{[0,\infty)} + (A_1 E - D_1) w_{[0,\infty)}\|_\infty \leq \|A_1\| c_w + \|A_1 E - D_1\| c_w =: c_{w2}$. By BIBS property of the $\bar{z}$ dynamics, $\exists c_{c1} \in \overline{\mathbb{R}_+}$, which depends only on c_{w1}, such that $\|\eta_{1[0,\infty)}\|_\infty \leq c_{c1}$. By the minimum phase property of S, $\exists c_{c2} \in \overline{\mathbb{R}_+}$, which depends only on c_w and c_{w2}, such that $\|\eta_{2[0,\infty)}\|_\infty \leq c_{c2}$. By the boundedness of solution to $\hat{z}$ dynamics, $\exists c_{c3} \in \overline{\mathbb{R}_+}$, which depends only on c_w, such that $\|\eta_{3[0,\infty)}\|_\infty \leq c_{c3}$. This implies that $\|\lambda_{z[0,\infty)}\|_\infty \leq c_{c1} + c_{c2} + c_{c3} =: c_c$. Hence, the composite system of S in tandem with S_1 is minimum phase with respect to $\mathbb{R}^m \times \mathcal{D}_0$ and $\mathcal{W}_d$.

This completes the proof of the theorem. ∎

Finally, we prove another interesting result.

Proposition 2: Consider the MIMO LTI system S and assume that S admits vector relative degree $(r_1, \ldots, r_m)$, $r_i \in \mathbb{Z}_+$, $i = 1, \ldots, m$. Then, the composite system (8) of S in tandem with S_1 admits vector relative degree $(r_1 + 1, \ldots, r_m + 1)$ if the matrix A_1 is diagonal.

Proof: By the vector relative degree assumption, we have $\mathbf{0}_{1\times p} = F_{i,:} = C_{i,:}B = \cdots = C_{i,:}A^{r_i - 2}B$, where $F_{i,:}$ and $C_{i,:}$ denote the ith row of the matrices F and C, respectively, $H_{i,:} := C_{i,:}A^{r_i-1}B \neq \mathbf{0}_{1\times p}$, ($H_{i,:} := F_{i,:}$ if $r_i = 0$) $i = 1, \ldots, m$, and $H := \begin{bmatrix} H_{1,:} \\ \vdots \\ H_{m,:} \end{bmatrix}$ is of full row rank. For the composite system (8), we have, $i = 1, \ldots, m$, $\bar{F}_{i,:} = \mathbf{0}_{1\times p}$, $\bar{C}_{i,:}\bar{B} = \boldsymbol{e}'_{m,i}F = F_{i,:} = \mathbf{0}_{1\times p}$, $\ldots$, $\bar{C}_{i,:}\bar{A}^{r_i-1}\bar{B} = \bar{C}_{i,:}\begin{bmatrix} A_1^{r_i-1} & \sum_{j=0}^{r_i-2} A_1^{r_i-2-j} CA^j \\ \mathbf{0}_{n\times m} & A^{r_i-1} \end{bmatrix}\bar{B} = \boldsymbol{e}'_{m,i}A_1^{r_i-1}F + \boldsymbol{e}'_{m,i}(\sum_{j=0}^{r_i-2} A_1^{r_i-2-j}CA^j)B = a_{ii}^{r_i-1}F_{i,:} + \sum_{j=0}^{r_i-2} a_{ii}^{r_i-2-j}C_{i,:}A^jB = \mathbf{0}_{1\times p}$, and $\bar{C}_{i,:}\bar{A}^{r_i}\bar{B} = \bar{C}_{i,:}\begin{bmatrix} A_1^{r_i} & \sum_{j=0}^{r_i-1} A_1^{r_i-1-j}CA^j \\ \mathbf{0}_{n\times m} & A^{r_i} \end{bmatrix}\bar{B} = \boldsymbol{e}'_{m,i}A_1^{r_i}F + \boldsymbol{e}'_{m,i}(\sum_{j=0}^{r_i-1} A_1^{r_i-1-j}CA^j)B = a_{ii}^{r_i}F_{i,:} + \sum_{j=0}^{r_i-1} a_{ii}^{r_i-1-j}C_{i,:}A^jB = C_{i,:}A^{r_i-1}B = H_{i,:} \neq \mathbf{0}_{1\times p}$, where $\bar{F}_{i,:}$ and $\bar{C}_{i,:}$ are the ith row of matrices $\bar{F}$ and $\bar{C}$, respectively, and $A_1 := \begin{bmatrix} a_{11} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & a_{mm} \end{bmatrix}$. This shows that the composite system (8) admits vector relative degree $(r_1 + 1, \ldots, r_m + 1)$. This completes the proof of the proposition. ∎

IV. Conclusions

In this paper, we proved that, for a MIMO LTI system S, attaching a dynamic/static controller without feedback will not decrease the dimension of the extended zero dynamics. Then, we proved that the composite system that comprises of a right invertible MIMO LTI system S in tandem with an integration block on the output of the system admits an extended zero dynamics that is of the same order as the one for S, and the composite system remains minimum phase when S is so to begin with. Finally, we proved that the composite system consisting of a MIMO LTI system S with vector relative degree $(r_1, \ldots, r_m)$, where $m \in \mathbb{Z}_+$ is the dimension of the output, in tandem with a diagonal integration block on the output of S admits vector relative degree $(r_1 + 1, \ldots, r_m + 1)$.

Future research along this direction lies in study of minimum phase properties of interconnected LTI systems, which are more general than the ones discussed in this paper.

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