An Extended Zero Dynamics Canonical Form for Finite-Dimensional Continuous-Time Parallel Interconnected Square MIMO LTI Systems ⋆

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Abstract: In this paper, we have obtained the extended zero-dynamics canonical form for a class of square MIMO LTI systems comprised of multiple square MIMO LTI subsystems in parallel interconnection satisfying an interconnection property. We assume that each subsystem has already been analyzed and extended to admit uniform vector relative degree (and has uniform observability indices), and thus is suitable for the design of robust adaptive controllers. We prescribe an interconnection property under which the composite system (without any further modification or extension) admits the extended zero-dynamics canonical form even though it does not have uniform vector relative degree. We further show that such a composite system, if connected in tandem with a diagonal integration block on the output will yield a system that also admits the extended zero-dynamics canonical form. This will allow a centralized robust adaptive controller design for the composite square MIMO LTI system if the composite system can be shown to be minimum phase.

Keywords: Minimum phase, zero dynamics canonical form, extended zero dynamics, extended zero dynamics canonical form, observability indices.

1. INTRODUCTION

The minimum phase concept for linear systems is crucial for the generalization of robust adaptive control system design for finite-dimensional continuous-time SISO LTI systems to that for finite-dimensional continuous-time MIMO LTI systems (see Pan and Başar (2000, 2018); Başar and Pan (2020); Pan and Başar (2023); Pan et al. (2023); Başar and Pan (2024)). In robust adaptive control for SISO systems (Pan and Başar, 2000) it has been observed that key canonical forms for the underlying system are the observer canonical form and the extended zero-dynamics canonical form. For square MIMO LTI systems, the zero-dynamics canonical form exists if there exists a vector relative degree for the system. This zero-dynamic canonical form then reveals the extended zero dynamics for the system. However, the extended zero-dynamics canonical form exists for general square MIMO LTI systems under a more restrictive assumption: the system must admit uniform vector relative degree (Başar and Pan, 2020, 2024). This means that one must extend the square MIMO LTI system to admit uniform vector relative degree before attempting to design a robust adaptive controller. These extra steps of extension lead to a larger system order and therefore a more complicated adaptive controller, and it does not allow for an easy expansion of the system when additional subsystems are incorporated into the composite system.

In this paper, we have obtained the extended zero-dynamics canonical form for a class of square MIMO LTI systems that is comprised of multiple square MIMO LTI subsystems in parallel interconnection satisfying an interconnection property. We assume that each subsystem has already been analyzed and extended to admit uniform vector relative degree (and has uniform observability indices, see Chen (1984)). We have multiple such subsystems parallel-interconnected to form a composite system, where the composite system admits vector relative degree but not uniform vector relative degree. We prescribe an interconnection property, under which the composite system (without any further modification or extension) admits

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the extended zero-dynamics canonical form even though it does not have uniform vector relative degree. Thus, the composite system would be in a form suitable for robust adaptive control design with nonuniform vector relative degree if it is further minimum phase according to Başar and Pan (2024). The interconnection property we prescribe is that for each subsystem i, the connections from subsystem j, $j \neq i$, satisfy the properties that the relative degree from each component of $\boldsymbol{y}_j$ to each component of $\boldsymbol{y}_i$ is greater than $\max\{0, r_i - r_j\}$, where r_i and r_j are the uniform vector relative degrees for the ith subsystem and the jth subsystem, respectively, and the relative degree from each component of $\boldsymbol{u}_j$ to each component of $\boldsymbol{y}_i$ is greater than $\max\{r_i, r_j\}$. We further show that such a composite system, if connected in tandem with a diagonal integration block on the output, will yield a system that also admits the extended zero-dynamics canonical form. Thus, when a number of subsystems are to be incorporated into a robust adaptive control system, we just need to make sure that these subsystems are themselves with uniform vector relative degree (and have uniform observability indices), and the interconnections of these subsystems and those of the original system are compatible, i. e., they satisfy the interconnection property. Then, the (centralized) robust adaptive controller can be redesigned and applied to the larger system without requiring any changes in the subsystems if the composite system is minimum phase according to Başar and Pan (2024).

The balance of the paper is as follows. In the next section, we introduce the notations used in the paper. In Section 3, we prove the existence of the extended zero dynamics canonical form for a class of square MIMO LTI systems which is the composite system of multiple square MIMO LTI systems in parallel interconnection further satisfying the interconnection property. The availability of the extended zero dyanmics canonical form for this class of systems is crucial for the robustness proof of adaptive controllers. In Section 4, we further show that such a composite system, if connected in tandem with a diagonal integration block on the output will yield a system that also admits the extended zero-dynamics canonical form (EZDCF). The paper ends with some concluding remarks in Section 5.

2. NOTATIONS

We let $\mathbb{R}$ denote the real line; $\mathbb{N}$ be the set of natural numbers; and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Unless otherwise specified, all signals, constants, and matrices are real. For any matrix A, A' denotes its transpose. For $n \in \mathbb{Z}_+$, I_n denotes the $n \times n$-dimensional identity matrix. For $n \in \mathbb{Z}_+$ and $n \times n$-dimensional matrix A, we set $A^0 = I_n$. For any $m, n \in \mathbb{Z}_+$, $\mathbf{0}_{m \times n}$ denotes the $m \times n$-dimensional matrix whose elements are all zeros. We will denote constants or matrices of no specific interest or relevance to the analysis by $\star$. We will denote $m \times n$-dimensional matrices of no specific interest or relevance to the analysis by $\star_{m \times n}$. For any $n \in \mathbb{N}$ and any $a := (a_i)_{i=1}^n, b := (b_i)_{i=1}^n \in \mathbb{R}^n$, $a \vee b \in \mathbb{R}^n$ denotes the vector $(\max\{a_i, b_i\})_{i=1}^n$; and $a \wedge b \in \mathbb{R}^n$ denotes the vector $(\min\{a_i, b_i\})_{i=1}^n$.

3. THE EXTENDED ZERO DYNAMICS CANONICAL FORM

In this section, we first recall the definition of the extended zero dynamics for a MIMO LTI system (Başar and Pan (2024)).

Consider a right-invertible MIMO LTI system (not necessarily square)

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} + D\boldsymbol{w}; \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathcal{D}_0 \tag{1a}$$
$$\boldsymbol{y} = C\boldsymbol{x} + F\boldsymbol{u} + E\boldsymbol{w} \tag{1b}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state, $n \in \mathbb{Z}_+$; $\boldsymbol{u} \in \mathbb{R}^p$ is the control input, $p \in \mathbb{Z}_+$; $\boldsymbol{y} \in \mathbb{R}^m$ is the output, $m \in \mathbb{Z}_+$; $\boldsymbol{w} \in \mathbb{R}^q$ is the disturbance input, $q \in \mathbb{Z}_+$; $\boldsymbol{x}_0 \in \mathcal{D}_0$, $\mathcal{D}_0 \subseteq \mathbb{R}^n$ is a subspace, $\boldsymbol{w}_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_q$ (Pan and Başar, 2018), A, B, D, C, F, and E are constant matrices of appropriate dimensions. The extended zero dynamics of (1) is defined by the maximal solution $K \in \mathbb{R}^{s \times n}$, $A_z \in \mathbb{R}^{s \times s}$, $A_{z1} \in \mathbb{R}^{s \times m}$ to the following matrix equations.

$$KA = A_z K + A_{z1} C \tag{2a}$$
$$A_{z1} F = KB \tag{2b}$$

where K is of full row rank such that s is maximal. Then, defining $\boldsymbol{x}_z := K\boldsymbol{x}$, it evolves according to

$$\dot{\boldsymbol{x}}_z = A_z \boldsymbol{x}_z + A_{z1}\boldsymbol{y} + (KD - A_{z1}E)\boldsymbol{w}; \qquad \boldsymbol{x}_z(0) = K\boldsymbol{x}_0 \in K(\mathcal{D}_0) \tag{3}$$

This is said to be the *extended zero dynamics* of (1). (Note that $s = 0$ is also a possible solution, which corresponding to the case when the extended zero dynamics is absent)

Then, we recall the canonical form that reveals the extended zero dynamics for square MIMO LTI systems with vector relative degree (See Isidori (1995) or Başar and Pan (2024)).

Lemma 1. Consider a square MIMO LTI system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} + D\boldsymbol{w}; \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathcal{D}_0 \tag{4a}$$
$$\boldsymbol{y} = C\boldsymbol{x} + F\boldsymbol{u} + E\boldsymbol{w} \tag{4b}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state, $n \in \mathbb{Z}_+$; $\boldsymbol{u} \in \mathbb{R}^m$ is the control input, $m \in \mathbb{Z}_+$; $\boldsymbol{y} \in \mathbb{R}^m$ is the output; $\boldsymbol{w} \in \mathbb{R}^q$ is the disturbance input, $q \in \mathbb{Z}_+$; $\boldsymbol{x}_0 \in \mathcal{D}_0$, $\mathcal{D}_0 \subseteq \mathbb{R}^n$ is a subspace, $\boldsymbol{w}_{[0,\infty)} \in \mathcal{W}_d$ of class $\mathcal{B}_q$, A, B, D, C, F, and E are constant matrices of appropriate dimensions.

Let the system admit vector relative degree $r_1, \ldots, r_m \in \{0, \ldots, n\}$ from $\boldsymbol{u}$ to $\boldsymbol{y}$, that is, $i = 1, \ldots, m$,

$$F_{i,:} = C_{i,:}B = \cdots = C_{i,:}A^{r_i - 2}B = \mathbf{0}_{1 \times m}$$

where $F_{i,:}$ and $C_{i,:}$ are the ith row vectors of the matrices F and C, respectively, and

$$\begin{bmatrix} C_{1,:}A^{r_1-1}B \\ \vdots \\ C_{m,:}A^{r_m-1}B \end{bmatrix} =: B_0$$

is an invertible matrix (for those $i = 1, \dots, m$ with $r_i = 0$, the corresponding row in B_0 is replaced by $F_{i,:}$). The matrix B_0 is said to be the *high frequency gain matrix*. Then, there exists an invertible matrix T_o such that, in

$$\bar{\boldsymbol{x}} := T_o^{-1}\boldsymbol{x} = \left[\boldsymbol{x}_z' \; \boldsymbol{x}_{1,1} \dots \boldsymbol{x}_{1,r_1} \dots \boldsymbol{x}_{m,1} \dots \boldsymbol{x}_{m,r_m}\right]'$$

coordinates, the system (4) admits the representation

$$\dot{\boldsymbol{x}}_z = A_z\boldsymbol{x}_z + \sum_{i=1}^{m} A_{z1,i}\boldsymbol{y}_i + D_z\boldsymbol{w} \tag{5a}$$

$$\dot{\boldsymbol{x}}_{i,j} = \boldsymbol{x}_{i,j+1} + D_{i,j}\boldsymbol{w}; \quad 1 \le i \le m \text{ with } r_i > 0, 1 \le j < r_i \tag{5b}$$

$$\dot{\boldsymbol{x}}_{i,r_i} = A_i\bar{\boldsymbol{x}} + C_{i,:}A^{r_i-1}B\boldsymbol{u} + D_{i,r_i}\boldsymbol{w}; \quad 1 \le i \le m \text{ with } r_i > 0 \tag{5c}$$

$$\boldsymbol{y}_i = \boldsymbol{x}_{i,1} + E_{i,:}\boldsymbol{w}; \; 1 \le i \le m \text{ with } r_i > 0 \tag{5d}$$

$$\boldsymbol{y}_i = \bar{C}_{i,:}\bar{\boldsymbol{x}} + F_{i,:}\boldsymbol{u} + E_{i,:}\boldsymbol{w}; \; 1 \le i \le m \text{ with } r_i = 0 \tag{5e}$$

where $\boldsymbol{x}_z \in \mathbb{R}^{n-\sum_{i=1}^{m} r_i}$; $\boldsymbol{x}_{i,j} \in \mathbb{R}$, $1 \le i \le m$ with $r_i > 0$, $1 \le j \le r_i$. (5) is called the *zero dynamics canonical form* of system (4). (Note that here (5) is not the extended zero dynamics canonical form.) The dynamics (5a) is said to be the *extended zero dynamics* of system (4).

When the vector relative degree $(r_1, \dots, r_m)$ is uniform, then Başar and Pan (2024) further defines the extended zero dynamics canonical form for (4) in four possible cases depending on the value $r = r_1 = \dots = r_m$, m, and n.

In this paper, we consider a special class of square MIMO LTI systems formed as the parallel interconnected square MIMO LTI systems as depicted in Figure 1. (For brevity, the figure only includes two interconnected subsystems, but we consider here an arbitrary number, p, of such interconnected subsystems.)

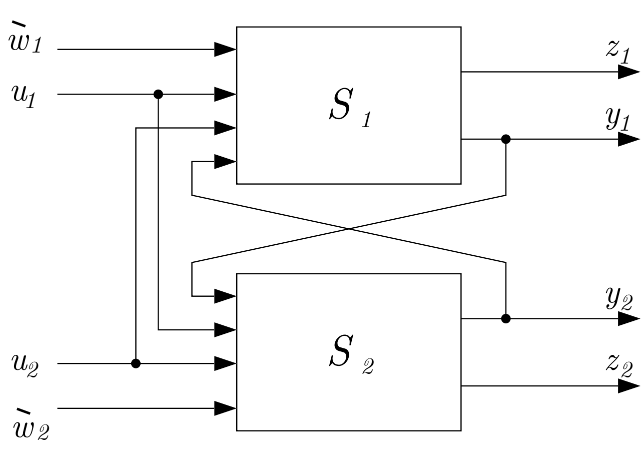

Fig. 1. Two parallel interconnected subsystems.

We assume that

Assumption 1. Each subsystem S_i, $i = 1, \dots, p$, is a finite-dimensional continuous-time square MIMO LTI system of order $n_i \ge 0$ and with uniform vector relative degree $0 \le r_i \le \frac{n_i}{m_i}$ from $\boldsymbol{u}_i$ to $\boldsymbol{y}_i$, where $\boldsymbol{u}_i$ and $\boldsymbol{y}_i$ are $m_i \in \mathbb{N}$ dimensional.

We assume the following interconnection properties:

Assumption 2. (Interconnection Property). Fix any $i = 1, \dots, p$, for subsystem S_i, the relative degree from each component of $\boldsymbol{y}_j$, $j \ne i$ to each component of $\boldsymbol{y}_i$ is greater than $0 \vee (r_i - r_j)$; and the relative degree from each component of $\boldsymbol{u}_j$, $j \ne i$, to each component of $\boldsymbol{y}_i$ is greater than $r_i \vee r_j$, $j = 1, \dots, p$.

It is straightforward to verify that the composite system S with input $\boldsymbol{u} := (\boldsymbol{u}_1, \dots, \boldsymbol{u}_p)$ and output $\boldsymbol{y} := (\boldsymbol{y}_1, \dots, \boldsymbol{y}_p)$ admits vector relative degree $(\underbrace{r_1, \dots, r_1}_{m_1-\text{times}}, \dots, \underbrace{r_p, \dots, r_p}_{m_p-\text{times}})$. Thus, by Lemma 1, for the composite system S, we have the following zero dynamics canonical form:

$$\dot{\boldsymbol{x}}_z = A_z\boldsymbol{x}_z + \sum_{i=1}^{p} A_{z1,i}\boldsymbol{y}_i + D_z\boldsymbol{w} \tag{6a}$$

$$\dot{\boldsymbol{x}}_{i,j} = \boldsymbol{x}_{i,j+1} + D_{i,j}\boldsymbol{w}; \quad 1 \le i \le p \text{ with } r_i > 0, 1 \le j < r_i \tag{6b}$$

$$\dot{\boldsymbol{x}}_{i,r_i} = A_i\boldsymbol{x} + B_{0,i}\boldsymbol{u}_i + D_{i,r_i}\boldsymbol{w}; \quad 1 \le i \le p \text{ with } r_i > 0 \tag{6c}$$

$$\boldsymbol{y}_i = \boldsymbol{x}_{i,1} + E_i\boldsymbol{w}; \; 1 \le i \le p \text{ with } r_i > 0 \tag{6d}$$

$$\boldsymbol{y}_i = C_i\boldsymbol{x} + F_i\boldsymbol{u}_i + E_i\boldsymbol{w}; \; 1 \le i \le p \text{ with } r_i = 0 \tag{6e}$$

where $\boldsymbol{x}_z \in \mathbb{R}^{\sum_{i=1}^{p} n_i - \sum_{i=1}^{p} m_i r_i}$; $\boldsymbol{y}_i \in \mathbb{R}^{m_i}$, $i = 1, \dots, p$; $\boldsymbol{x}_{i,j} \in \mathbb{R}^{m_i}$, $i = 1, \dots, p$ with $r_i > 0$, $j = 1, \dots, r_i$, $B_{0,i}$ is invertible, $i = 1, \dots, p$ with $r_i > 0$; and F_i is invertible, $i = 1, \dots, p$ with $r_i = 0$; $\boldsymbol{x} = (\boldsymbol{x}_z, \boldsymbol{x}_{1,1}, \dots, \boldsymbol{x}_{1,r_1}, \dots, \boldsymbol{x}_{p,1}, \dots, \boldsymbol{x}_{p,r_p})$. (6) is the *zero dynamics canonical form* of system S. (Note that here (6) is not the extended zero dynamics canonical form.) The dynamics (6a) is the *extended zero dynamics* of system S. Without loss of generality, assume that

Assumption 3. The uniform vector relative degrees are ordered in the nondecreasing fashion: $r_1 \le r_2 \le \dots \le r_p$.

By Assumption 2, we have C_i, $i = 1, \dots, p$ with $r_i = 0$, has all zero elements multiplying $\boldsymbol{x}_{j,1}, \dots, \boldsymbol{x}_{j,r_j}$, $j = 1, \dots, p$ with $r_j > 0$. Therefore, C_i has nonzero elements only multiplying $\boldsymbol{x}_z$. By Assumption 2, we have A_i, $i = 1, \dots, p$ with $r_i > 0$, has all zero elements multiplying $\boldsymbol{x}_{j,r_i+1}, \dots, \boldsymbol{x}_{j,r_j}$, $j = 1, \dots, p$ with $r_j > r_i$.

Now, consider the dynamics of

$$(\boldsymbol{x}_{1,1}, \dots, \boldsymbol{x}_{1,r_1}, \dots, \boldsymbol{x}_{p,1}, \dots, \boldsymbol{x}_{p,r_p})$$

It has the following structure:

$$\dot{\bar{\boldsymbol{x}}}_{i,j} = \bar{\boldsymbol{x}}_{i,j+1} + \bar{D}_{i,j}\boldsymbol{w}; \quad i = 1, \dots, \bar{p}, j = 1, \dots, \bar{r}_i - 1 \tag{7a}$$

$$\dot{\bar{\boldsymbol{x}}}_{i,\bar{r}_i} = \bar{A}_{iz}\boldsymbol{x}_z + \sum_{j=1}^{\bar{p}} \sum_{l=1}^{\bar{r}_j \wedge \bar{r}_i} \bar{A}_{i,j,l}\bar{\boldsymbol{x}}_{j,l} + \bar{B}_{0,i}\bar{\boldsymbol{u}}_i + \bar{D}_{i,\bar{r}_i}\boldsymbol{w}; \quad (7b)$$

$$\bar{\boldsymbol{y}}_i = \bar{\boldsymbol{x}}_{i,1} + \bar{E}_i\boldsymbol{w}; \quad i = 1, \ldots, \bar{p} \quad (7c)$$

where $\bar{p}$ is equal to the number of distinct r_i, $i = 1, \ldots, p$, that are not zeros, which forms the set $\{\bar{r}_1, \ldots, \bar{r}_{\bar{p}}\}$; $l_1, \ldots, l_{\bar{p}}$ is defined by $r_{l_1} = 0 < \bar{r}_1 := r_{l_1+1} = \cdots = r_{l_2} < \bar{r}_2 := r_{l_2+1} = \cdots = r_{l_3} < \cdots < \bar{r}_{\bar{p}} := r_{l_{\bar{p}}+1} = \cdots = r_p$, $l_{\bar{p}+1} = p$; (for notational consistency, we define $r_0 := 0$;) $\bar{\boldsymbol{x}}_{i,j} := (\boldsymbol{x}_{l_i+1,j}, \ldots, \boldsymbol{x}_{l_{i+1},j})$, $i = 1, \ldots, \bar{p}$, $j = 1, \ldots, \bar{r}_i$, $\bar{\boldsymbol{u}}_i := (\boldsymbol{u}_{l_i+1}, \ldots, \boldsymbol{u}_{l_{i+1}})$, and $\bar{B}_{0,i} :=$ block diagonal $\left(B_{0,l_i+1}, \ldots, B_{0,l_{i+1}}\right)$, $\bar{\boldsymbol{y}}_i := (\boldsymbol{y}_{l_i+1}, \ldots, \boldsymbol{y}_{l_{i+1}})$, $i = 1, \ldots, \bar{p}$.

For the system (7), $\boldsymbol{x}_z$ and $\boldsymbol{w}$ are considered inputs into the system. By Lemma 1 of Pan and Başar (2024a), this system (7) is observable with observability indices

$$(\underbrace{\bar{r}_1, \ldots, \bar{r}_1}_{\bar{m}_1-\text{times}}, \ldots, \underbrace{\bar{r}_{\bar{p}}, \ldots, \bar{r}_{\bar{p}}}_{\bar{m}_{\bar{p}}-\text{times}})$$

where $\bar{m}_i := \sum_{j=l_i+1}^{l_{i+1}} m_j$, $i = 1, \ldots, \bar{p}$. As we had done in Lemma 2 of Başar and Pan (2024), we will transform the system (7) into observer canonical form. By the proof of Lemma 1 of Pan and Başar (2024a), we note that the noninterweaved version of the matrix $Q = I_{\sum_{i=1}^{\bar{p}} \bar{r}_i\bar{m}_i}$. Thus, we may form the matrix S as in the proof of Lemma 1 of Pan and Başar (2024a) (noninterweaved version):

$$S = \begin{bmatrix} \bar{M}_{11} & \cdots & \bar{M}_{1\bar{p}} \\ \vdots & \ddots & \vdots \\ \bar{M}_{\bar{p}1} & \cdots & \bar{M}_{\bar{p}\bar{p}} \end{bmatrix}$$

where $\bar{M}_{ii}$, $i = 1, \ldots, \bar{p}$, is a $\bar{r}_i\bar{m}_i \times \bar{r}_i\bar{m}_i$-dimensional matrix of the form:

$$\bar{M}_{ii} = \begin{bmatrix} I_{\bar{m}_i} & \mathbf{0} & \cdots & \mathbf{0} \\ \star & I_{\bar{m}_i} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \star & \cdots & \star & I_{\bar{m}_i} \end{bmatrix}$$

and $\bar{M}_{ij}$, $i = 1, \ldots, \bar{p}-1$, $j = i+1, \ldots, \bar{p}$, is a $\bar{r}_i\bar{m}_i \times \bar{r}_j\bar{m}_j$-dimensional matrix of the form:

$$\bar{M}_{ij} = \begin{bmatrix} \mathbf{0}_{\bar{m}_i\times\bar{m}_j} & \cdots & \cdots & \cdots & \cdots & \mathbf{0}_{\bar{m}_i\times\bar{m}_j} \\ \star_{\bar{m}_i\times\bar{m}_j} & \mathbf{0}_{\bar{m}_i\times\bar{m}_j} & \cdots & \cdots & \cdots & \mathbf{0}_{\bar{m}_i\times\bar{m}_j} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \star_{\bar{m}_i\times\bar{m}_j} & \cdots & \star_{\bar{m}_i\times\bar{m}_j} & \mathbf{0}_{\bar{m}_i\times\bar{m}_j} & \cdots & \mathbf{0}_{\bar{m}_i\times\bar{m}_j} \end{bmatrix}$$

and $\bar{M}_{ij}$, $i = 2, \ldots, \bar{p}$, $j = 1, \ldots, i-1$, is a $\bar{r}_i\bar{m}_i \times \bar{r}_j\bar{m}_j$-dimensional matrix of the form:

$$\bar{M}_{ij} = \begin{bmatrix} \mathbf{0}_{\bar{m}_i\times\bar{m}_j} & \mathbf{0}_{\bar{m}_i\times\bar{m}_j} & \cdots & \mathbf{0}_{\bar{m}_i\times\bar{m}_j} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0}_{\bar{m}_i\times\bar{m}_j} & \vdots & \vdots & \vdots \\ \star_{\bar{m}_i\times\bar{m}_j} & \mathbf{0}_{\bar{m}_i\times\bar{m}_j} & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \star_{\bar{m}_i\times\bar{m}_j} & \cdots & \star_{\bar{m}_i\times\bar{m}_j} & \mathbf{0}_{\bar{m}_i\times\bar{m}_j} \end{bmatrix}$$

We have the following result.

Lemma 2. The matrix S is invertible and S^{-1} admits the same structure as S.

Proof. We will show that S is always invertible and S^{-1} admits the same structure as S using mathematical induction on $\bar{p}$.

1° Consider the case $\bar{p} = 1$. The result is obvious.

2° Assume that the result holds when $\bar{p} = k \in \mathbb{N}$.

3° Consider the case when $\bar{p} = k+1 \in \{2, 3, \ldots\}$. Denote $\bar{S} := \begin{bmatrix} \bar{M}_{11} & \cdots & \bar{M}_{1k} \\ \vdots & \ddots & \vdots \\ \bar{M}_{k1} & \cdots & \bar{M}_{kk} \end{bmatrix}$ and then $\left[\begin{array}{c|c} \bar{S} & \begin{matrix} \bar{M}_{1\bar{p}} \\ \vdots \\ \bar{M}_{\bar{p}-1,\bar{p}} \end{matrix} \\ \hline \bar{M}_{\bar{p}1} \cdots \bar{M}_{\bar{p},\bar{p}-1} & \bar{M}_{\bar{p}\bar{p}} \end{array}\right] = S$. By 2°, $\bar{S}$ is invertible and $\bar{S}^{-1}$ admits the same form as $\bar{S}$. By 1°, $\bar{M}_{\bar{p}\bar{p}}^{-1}$ exists and admits the same form as $\bar{M}_{\bar{p}\bar{p}}$. By Matrix Inversion Lemma, we have $S^{-1} = \begin{bmatrix} T_{11}^{-1} & T_{12} \\ T_{21} & T_{22}^{-1} \end{bmatrix}$ when $\bar{S}$, $\bar{M}_{\bar{p}\bar{p}}$ are invertible, and T_{11} and T_{22} are invertible, where $T_{11} = \bar{S} - \begin{bmatrix} \bar{M}_{1\bar{p}} \\ \vdots \\ \bar{M}_{\bar{p}-1,\bar{p}} \end{bmatrix} \bar{M}_{\bar{p}\bar{p}}^{-1} \left[\bar{M}_{\bar{p}1} \cdots \bar{M}_{\bar{p},\bar{p}-1}\right]$, $T_{21} = -\bar{M}_{\bar{p}\bar{p}}^{-1} \left[\bar{M}_{\bar{p}1} \cdots \bar{M}_{\bar{p},\bar{p}-1}\right] T_{11}^{-1}$, $T_{22} = -\left[\bar{M}_{\bar{p}1} \cdots \bar{M}_{\bar{p},\bar{p}-1}\right] \cdot \bar{S}^{-1} \begin{bmatrix} \bar{M}_{1\bar{p}} \\ \vdots \\ \bar{M}_{\bar{p}-1,\bar{p}} \end{bmatrix} + \bar{M}_{\bar{p}\bar{p}}$, and $T_{12} = -\bar{S}^{-1} \begin{bmatrix} \bar{M}_{1\bar{p}} \\ \vdots \\ \bar{M}_{\bar{p}-1,\bar{p}} \end{bmatrix} T_{22}^{-1}$.

Now, based on the structures preceding the lemma, we can easily show that T_{11} admits the same structure as $\bar{S}$ and therefore invertible, by 2°. Then, T_{11}^{-1} admits the same structure as $\bar{S}^{-1}$, which is the same structure as $\bar{S}$, by 2°. We can also conclude that T_{22} admits the same structure as $\bar{M}_{\bar{p}\bar{p}}$. By 1°, T_{22}^{-1} exists and admits the same structure as $\bar{M}_{\bar{p}\bar{p}}$. Furthermore, T_{12} admits the same structure as $\begin{bmatrix} \bar{M}_{1\bar{p}} \\ \vdots \\ \bar{M}_{\bar{p}-1,\bar{p}} \end{bmatrix}$ and T_{21} admits the same structure as $\left[\bar{M}_{\bar{p}1} \cdots \bar{M}_{\bar{p},\bar{p}-1}\right]$. Thus, S^{-1} admits the same structure as S.

This completes the induction process and therefore the proof of the lemma. □

By Lemma 1 of Pan and Başar (2024a), in the coordinates of

$$\begin{aligned} &(\check{\boldsymbol{x}}_{1,1}, \ldots, \check{\boldsymbol{x}}_{1,\bar{r}_1}, \ldots, \check{\boldsymbol{x}}_{\bar{p},1}, \ldots \check{\boldsymbol{x}}_{\bar{p},\bar{r}_{\bar{p}}}) \\ &:= S^{-1}(\bar{\boldsymbol{x}}_{1,1}, \ldots, \bar{\boldsymbol{x}}_{1,\bar{r}_1}, \ldots, \bar{\boldsymbol{x}}_{\bar{p},1}, \ldots \bar{\boldsymbol{x}}_{\bar{p},\bar{r}_{\bar{p}}}) \end{aligned}$$

the system (7) admits the observer canonical form representation:

$$\dot{\check{\boldsymbol{x}}}_{i,j} = \sum_{l=1}^{\bar{p}} \check{A}_{i,j,l}\check{\boldsymbol{x}}_{l,1} + \check{\boldsymbol{x}}_{i,j+1} + \check{D}_{i,j}\boldsymbol{w}; \quad (8a)$$

$$i=1,\dots,\bar{p}, j=1,\dots,\bar{r}_i-1$$

$$\dot{\check{\boldsymbol{x}}}_{i,\bar{r}_i} = \bar{A}_{iz}\boldsymbol{x}_z + \sum_{l=1}^{\bar{p}} \check{A}_{i,\bar{r}_i,l}\check{\boldsymbol{x}}_{l,1} + \bar{B}_{0,i}\bar{\boldsymbol{u}}_i + \check{D}_{i,\bar{r}_i}\boldsymbol{w}; \quad (8b)$$

$$\bar{\boldsymbol{y}}_i = \check{\boldsymbol{x}}_{i,1} + \bar{E}_i\boldsymbol{w}; \quad i=1,\dots,\bar{p} \quad (8c)$$

where $\check{\boldsymbol{x}}_{i,j}$ is of $\bar{m}_i$-dimensional, $i=1,\dots,\bar{p}$, $j=1,\dots,\bar{r}_i$; $\check{A}_{i,j,l}=\mathbf{0}_{\bar{m}_i\times\bar{m}_l}$, if $\bar{r}_i-\bar{r}_l-j\geq 0$, $i=1,\dots,\bar{p}$, $j=1,\dots,\bar{r}_i$, $l=1,\dots,\bar{p}$; and we have made use of the structure of S and S^{-1} in the above formulas.

Summarizing the preceding, we state the following result.

Proposition 3. Consider the finite-dimensional continuous-time square MIMO LTI system which is composed of $p\in\mathbb{N}$ finite-dimensional continuous-time square MIMO LTI systems in parallel configuration as illustrated in Figure 1 (which is for the special case $p=2$). We assume that Assumptions 1, 2, and 3 hold for the interconnected systems. Then, the composite system S admits state space representation (6) by Lemma 1. Furthermore, the system admits the following *extended zero dynamic canonical form*:

$$\dot{\boldsymbol{x}}_z = A_z\boldsymbol{x}_z + \sum_{i=1}^{l_1} A_{z1,i}\boldsymbol{y}_i + \sum_{i=1}^{\bar{p}} \check{A}_{z1,i}\bar{\boldsymbol{y}}_i + D_z\boldsymbol{w} \quad (9a)$$

$$\dot{\check{\boldsymbol{x}}}_{i,j} = \sum_{l=1}^{\bar{p}} \check{A}_{i,j,l}\check{\boldsymbol{x}}_{l,1} + \check{\boldsymbol{x}}_{i,j+1} + \check{D}_{i,j}\boldsymbol{w}; \quad (9b)$$

$$i=1,\dots,\bar{p}, j=1,\dots,\bar{r}_i-1$$

$$\dot{\check{\boldsymbol{x}}}_{i,\bar{r}_i} = \bar{A}_{iz}\boldsymbol{x}_z + \sum_{l=1}^{\bar{p}} \check{A}_{i,\bar{r}_i,l}\check{\boldsymbol{x}}_{l,1} + \bar{B}_{0,i}\bar{\boldsymbol{u}}_i + \check{D}_{i,\bar{r}_i}\boldsymbol{w}; \quad (9c)$$

$$\bar{\boldsymbol{y}}_i = \check{\boldsymbol{x}}_{i,1} + \bar{E}_i\boldsymbol{w}; \quad i=1,\dots,\bar{p} \quad (9d)$$

$$\boldsymbol{y}_i = C_{iz}\boldsymbol{x}_z + F_i\boldsymbol{u}_i + E_i\boldsymbol{w}; \ 1\leq i\leq l_1 \quad (9e)$$

where $\bar{p}$ is equal to the number of distinct r_i, $i=1,\dots,p$, that are not zeros, which forms the set $\{\bar{r}_1,\dots,\bar{r}_{\bar{p}}\}$; $l_1,\dots,l_{\bar{p}+1}$ is defined by $r_0=\cdots=r_{l_1}=0<\bar{r}_1:=r_{l_1+1}=\cdots=r_{l_2}<\bar{r}_2:=r_{l_2+1}=\cdots=r_{l_3}<\cdots<\bar{r}_{\bar{p}}:=r_{l_{\bar{p}}+1}=\cdots=r_p$, $l_{\bar{p}+1}:=p$; (for notational consistency, we define $r_0:=0$;) $\bar{\boldsymbol{u}}_i:=(\boldsymbol{u}_{l_i+1},\dots,\boldsymbol{u}_{l_{i+1}})$, and $\bar{B}_{0,i}:=$ block diagonal $\left(B_{0,l_i+1},\dots,B_{0,l_{i+1}}\right)$, which is invertible; $\bar{\boldsymbol{y}}_i:=(\boldsymbol{y}_{l_i+1},\dots,\boldsymbol{y}_{l_{i+1}})$, $i=1,\dots,\bar{p}$; $\boldsymbol{x}_z\in\mathbb{R}^{\sum_{i=1}^p n_i-\sum_{i=1}^p m_i r_i}$; $\check{\boldsymbol{x}}_{i,j}\in\mathbb{R}^{\bar{m}_i}=\mathbb{R}^{\sum_{k=l_i+1}^{l_{i+1}} m_i}$, $i=1,\dots,\bar{p}$, $j=1,\dots,\bar{r}_i$; F_i is invertible, $i=1,\dots,l_1$; and $\check{A}_{i,j,l}=\mathbf{0}_{\bar{m}_i\times\bar{m}_l}$, if $\bar{r}_i-\bar{r}_l-j\geq 0$, $i=1,\dots,\bar{p}$, $j=1,\dots,\bar{r}_i$, $l=1,\dots,\bar{p}$.

4. EZDCF FOR PARALLEL INTERCONNECTED SYSTEMS IN TANDEM WITH A DIAGONAL INTEGRATION BLOCK

In this section, we will consider the system studied in Proposition 3 in tandem with a diagonal integration block $\bar{S}$:

$$\dot{\boldsymbol{\xi}} = \bar{A}\boldsymbol{\xi} + \boldsymbol{y} + \bar{D}\boldsymbol{w} \quad (10)$$

where $\boldsymbol{\xi}$ is the state of $\bar{S}$ and the output of the composite system $\hat{S}$ that consists of system S in tandem with $\bar{S}$, and $\boldsymbol{y}$ is the output of the system S; $\boldsymbol{w}$ is the disturbance input of the system S; and the matrix $\bar{A}$ is a diagonal matrix. By Proposition 2 of Pan and Başar (2024b), the system $\hat{S}$ admits vector relative degree $(\underbrace{r_1+1,\dots,r_1+1}_{m_1\text{-times}},\dots,\underbrace{r_p+1,\dots,r_p+1}_{m_p\text{-times}})$.

By Lemma 1, the composite system $\hat{S}$ admits the following zero dynamics canonical form:

$$\dot{\hat{\boldsymbol{x}}}_z = \hat{A}_z\hat{\boldsymbol{x}}_z + \sum_{i=1}^{p} \hat{A}_{z1,i}\boldsymbol{\xi}_i + \hat{D}_z\boldsymbol{w} \quad (11a)$$

$$\dot{\hat{\boldsymbol{x}}}_{i,j} = \hat{\boldsymbol{x}}_{i,j+1} + \hat{D}_{i,j}\boldsymbol{w}; \ 1\leq i\leq p, \ 1\leq j\leq r_i \quad (11b)$$

$$\dot{\hat{\boldsymbol{x}}}_{i,r_i+1} = \hat{A}_i\hat{\boldsymbol{x}} + \hat{B}_{0,i}\boldsymbol{u}_i + \hat{D}_{i,r_i+1}\boldsymbol{w}; \ 1\leq i\leq p \quad (11c)$$

$$\boldsymbol{\xi}_i = \hat{\boldsymbol{x}}_{i,1}; \ 1\leq i\leq p \quad (11d)$$

where $\hat{\boldsymbol{x}}_z\in\mathbb{R}^{\sum_{i=1}^p n_i-\sum_{i=1}^p m_i r_i}$; $\boldsymbol{\xi}_i\in\mathbb{R}^{m_i}$, $i=1,\dots,p$; $\hat{\boldsymbol{x}}_{i,j}\in\mathbb{R}^{m_i}$, $i=1,\dots,p$, $j=1,\dots,r_i+1$, $\hat{B}_{0,i}$ is invertible, $i=1,\dots,p$; $\boldsymbol{\xi}=(\boldsymbol{\xi}_1,\dots,\boldsymbol{\xi}_p)$; $\hat{\boldsymbol{x}}=(\hat{\boldsymbol{x}}_z,\hat{\boldsymbol{x}}_{1,1},\dots,\hat{\boldsymbol{x}}_{1,r_1+1},\dots,\hat{\boldsymbol{x}}_{p,1},\dots,\hat{\boldsymbol{x}}_{p,r_p+1})$. (11) is the *zero dynamics canonical form* of system $\hat{S}$. (Note that here (11) is not the extended zero dynamics canonical form.) The dynamics (11a) is the *extended zero dynamics* of composite system $\hat{S}$.

Based on (11d), (6), and (10), we have $\hat{\boldsymbol{x}}_{i,2}=\bar{A}_i\boldsymbol{\xi}_i+\boldsymbol{x}_{i,1}$, $1\leq i\leq p$ with $r_i>0$; and $\hat{A}_i\hat{\boldsymbol{x}}=\bar{A}_i\boldsymbol{\xi}_i+C_i\boldsymbol{x}$, $1\leq i\leq p$ with $r_i=0$; where $\bar{A}=$ block diagonal $\left(\bar{A}_1,\dots,\bar{A}_p\right)$. By Assumption 2, we have $\hat{A}_i$, $i=1,\dots,p$ with $r_i=0$, has all zero elements multiplying $\hat{\boldsymbol{x}}_{j,2},\dots,\hat{\boldsymbol{x}}_{j,r_j+1}$, $j=1,\dots,p$ with $r_j>0$. (Otherwise the relative degree from $\boldsymbol{u}_j$ to $\boldsymbol{y}_i$ will be less than or equal to $r_j=\max\{0,r_j\}=\max\{r_i,r_j\}$.) By a similar argument, we have $\hat{A}_i$, $i=1,\dots,p$ with $r_i>0$, has all zero elements multiplying $\hat{\boldsymbol{x}}_{j,r_i+2},\dots,\hat{\boldsymbol{x}}_{j,r_j+1}$, $j=1,\dots,p$ with $r_j>r_i$. (Otherwise the relative degree from $\boldsymbol{u}_j$ to $\boldsymbol{y}_i$ will be less than or equal to $r_j=\max\{r_i,r_j\}$.) Thus, the dynamics of $(\hat{\boldsymbol{x}}_{1,1},\dots,\hat{\boldsymbol{x}}_{1,r_1+1},\dots,\hat{\boldsymbol{x}}_{p,1},\dots,\hat{\boldsymbol{x}}_{p,r_p+1})$ has the following structure:

$$\dot{\hat{\boldsymbol{x}}}_{i,j} = \hat{\boldsymbol{x}}_{i,j+1} + \hat{D}_{i,j}\boldsymbol{w}; \ 1\leq i\leq p, \ 1\leq j\leq r_i \quad (12a)$$

$$\dot{\hat{\boldsymbol{x}}}_{i,r_i+1} = \hat{A}_{iz}\hat{\boldsymbol{x}}_z + \sum_{j=1}^{p}\sum_{l=1}^{r_i\wedge r_j+1} \hat{A}_{i,j,l}\hat{\boldsymbol{x}}_{j,l} + \hat{B}_{0,i}\boldsymbol{u}_i + \hat{D}_{i,r_i+1}\boldsymbol{w}; \ 1\leq i\leq p \quad (12b)$$

$$\boldsymbol{\xi}_i = \hat{\boldsymbol{x}}_{i,1}; \ 1\leq i\leq p \quad (12c)$$

where $\hat{\boldsymbol{x}}_z$ and $\boldsymbol{w}$ are considered inputs to the system. The dynamics (12) has the same structure as (7) (after lumping the states corresponding to the same $\bar{r}_i$, $i=1,\dots,\bar{p}$; $\bar{\hat{\boldsymbol{x}}}_{i,j}:=(\hat{\boldsymbol{x}}_{l,j})_{l=l_i+1}^{l_{i+1}}$; $i=1,\dots,\bar{p}$, $j=1,\dots,\bar{r}_i+1$; and $\bar{\boldsymbol{\xi}}_i:=(\boldsymbol{\xi}_l)_{l=l_i+1}^{l_{i+1}}$, $i=1,\dots,\bar{p}$). By the same argument as the last paragraph before Lemma 2 and Lemma 2, we have

(12) can be transformed into the observer canonical form by Lemma 1 of Pan and Başar (2024a).

Summarizing the preceding, we state the following result.

Proposition 4. Consider the finite-dimensional continuous-time square MIMO LTI system S which is composed of $p \in \mathbb{N}$ finite-dimensional continuous-time square MIMO LTI systems in parallel configuration as illustrated in Figure 1 (which is for the special case $p = 2$). We assume that Assumptions 1, 2, and 3 hold for the interconnected systems. Then, the composite system $\hat{S}$ consists of S in tandem with the diagonal integration block $\bar{S}$, (10), admits state space representation (11) by Lemma 1. Furthermore, the system admits the *extended zero dynamic canonical form*:

$$\dot{\hat{\boldsymbol{x}}}_z = \hat{A}_z \hat{\boldsymbol{x}}_z + \sum_{i=1}^{p} \hat{A}_{z1,i} \boldsymbol{\xi}_i + \hat{D}_z \boldsymbol{w} \tag{13a}$$

$$\dot{\tilde{\boldsymbol{x}}}_{i,j} = \sum_{l=1}^{p} \tilde{A}_{i,j,l} \tilde{\boldsymbol{x}}_{l,1} + \tilde{\boldsymbol{x}}_{i,j+1} + \tilde{D}_{i,j} \boldsymbol{w}; \quad i = 1, \ldots, p, j = 1, \ldots, r_i \tag{13b}$$

$$\dot{\tilde{\boldsymbol{x}}}_{i,r_i+1} = \hat{A}_{iz} \hat{\boldsymbol{x}}_z + \sum_{l=1}^{p} \tilde{A}_{i,r_i+1,l} \tilde{\boldsymbol{x}}_{l,1} + \hat{B}_{0,i} \boldsymbol{u}_i + \tilde{D}_{i,r_i+1} \boldsymbol{w}; \; i = 1, \ldots, p \tag{13c}$$

$$\boldsymbol{\xi}_i = \tilde{\boldsymbol{x}}_{i,1}; \quad i = 1, \ldots, p \tag{13d}$$

where $\hat{\boldsymbol{x}}_z \in \mathbb{R}^{\sum_{i=1}^{p} n_i - \sum_{i=1}^{p} m_i r_i}$; $\hat{B}_{0,i}$ is invertible; $i = 1, \ldots, p$; $\tilde{\boldsymbol{x}}_{i,j} \in \mathbb{R}^{m_i}$, $i = 1, \ldots, p$, $j = 1, \ldots, r_i + 1$.

5. CONCLUSIONS

In this paper, we have obtained the extended zero-dynamics canonical form for a class of square MIMO LTI systems comprised of multiple square MIMO LTI subsystems in parallel interconnection satisfying the interconnection property. Under the assumption that each subsystem has already been analyzed and extended to admit uniform vector relative degree (and has uniform observability indices), we have considered multiple such subsystems, parallel-interconnected to form a composite system which admits vector relative degree but not uniform vector relative degree. We have prescribed an interconnection property under which the composite system (without any further modification or extension) admits the extended zero-dynamics canonical form even though it does not have uniform vector relative degree. Thus, the composite system is in a form suitable for robust adaptive control design with nonuniform vector relative degree if it is further minimum phase according to Başar and Pan (2024). The interconnection property we have prescribed is one where for each subsystem i, the connections from subsystem j, $j \neq i$, satisfy the properties that the relative degree from each component of $\boldsymbol{y}_j$ to each component of $\boldsymbol{y}_i$ is greater than $\max\{0, r_i - r_j\}$, where r_i and r_j are the uniform vector relative degrees for the ith subsystem and the jth subsystem, respectively, and the relative degree from each component of $\boldsymbol{u}_j$ to each component of $\boldsymbol{y}_i$ is greater than $\max\{r_i, r_j\}$. We have further shown that such a composite system, if connected in tandem with a diagonal integration block on the output will yield a system that also admits the extended zero-dynamics canonical form. Thus, when a number of subsystems are to be incorporated into a robust adaptive control system, we just need to make sure that these subsystems are themselves with uniform vector relative degree (and have uniform observability indices), and the interconnections of these subsystems and those of the original system are compatible, i. e., they satisfy the interconnection property. Then, the (centralized) robust adaptive controller can be redesigned and applied to the larger system without requiring any changes in the subsystems if the composite system is minimum phase according to Başar and Pan (2024).

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